Derived equivalences from cohomological approximations, and mutations of $\Phi$-Yoneda algebras

Wei Hu, Steffen Koenig and Changchang Xi

Abstract

In this article, a new construction of derived equivalences is given. It relates different endomorphism rings and more generally cohomological endomorphism rings - including higher extensions - of objects in triangulated categories. These objects need to be connected by certain universal maps that are cohomological approximations and that exist in very general circumstances. The construction turns out to be applicable in a wide variety of situations, covering finite dimensional algebras as well as certain infinite dimensional algebras, Frobenius categories and $n$-Calabi-Yau categories.

1 Introduction

Derived equivalences have become increasingly important in representation theory, Lie theory and geometry. Examples are ranging from mirror symmetry over non-commutative geometry to the Kazhdan-Lusztig conjecture and to Broué’s conjecture for blocks of finite groups. In all of these situations, and in many others, derived equivalences that involve finite or infinite dimensional algebras are used. Derived equivalences between algebras, or rings, exist if and only if there exist suitable tilting complexes, as explained quite satisfactorily by Rickard’s Morita theory for derived categories of rings (see [21]). Derived equivalences have been shown to preserve many significant algebraic and geometric invariants and often to provide unexpected and useful new connections.

A crucial question in this context has, however, not yet received enough answers: How to construct derived equivalences between rings in a general setup?

A good answer - certainly not unique - to this question should be general, flexible and systematic and apply to a multitude of algebraic and geometric situations.

One well developed approach is based on the theory of tilting modules, building upon results by Happel [9]. Other answers use ring theoretic constructions, such as trivial extensions [22].

The aim of this article is to provide a rather different approach. The input of the technology developed here is a triple of objects $(X, M, Y)$ in a triangulated category. These objects are required to be related by certain universal maps (cohomological approximations - a new concept introduced here, continuing approximation theory of Auslander, Reiten and Smalø [1]) and some cohomological orthogonality conditions in degrees different from zero only. The output is a derived equivalence between cohomological endomorphism rings of $X \oplus M$ and of $M \oplus Y$.

The flexibility of the construction lies in the following features: We enhance endomorphism rings by higher extensions to produce cohomological endomorphism rings, broadening the classical concept of Yoneda extension algebras. Here, we can choose a set of cohomological degrees to define the cohomological endomorphism ring. Choosing degree zero only gives endomorphism rings in the usual sense - and then no orthogonality assumption is needed. Choosing all integers, or a suitable subset thereof (satisfying an associativity constraint), amplifies the concept of Yoneda extension algebras $\bigoplus \operatorname{Ext}^i(S, S)$. There is also some flexibility in the choice of $M$.

* Corresponding author. Email: xicc@bnu.edu.cn; Fax: 0086 10 58802136; Tel.: 0086 10 58808877.
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A special case of such a triple is given by any Auslander-Reiten triangle \( X \rightarrow M \rightarrow Y \) in a derived module category; this already indicates generality of the construction. Our assumptions are actually much more general and not limited to objects in derived categories of algebras.

A particular feature of the derived equivalences constructed by this method is that they also provide a very general mutation procedure, turning one ring into another one in a systematic way. Tilting theory has arisen as a far reaching extension of reflection functors for quivers. Under some assumptions, but not in general, it provides mutation procedures between two given quivers or algebras, both of which are endomorphism rings of tilting modules; in the case of quivers one may reflect at sink or source vertices. Mutations similar in style also have come up in various geometric situations. The theory of cluster categories, or more generally of Calabi-Yau categories, has extended reflections to a mutation procedure, which works for representations of quivers at all vertices. Such mutations fit into the present framework as well. There is, though, a new feature introduced by our approach: Reflection does not work in general in derived categories (of quivers or algebras). Therefore cluster theory passes to the cluster category, a ‘quotient’ of a derived category modulo the action of some functor; endomorphism rings are taken there. In contrast to this, the current approach always produces equivalences on the level of derived categories, not just of quotient categories; throughout we are considering derived equivalences between (cohomological) endomorphism rings or quotients thereof. In the case of quivers, this possibility of passing to quotient algebras allows mutation at an arbitrary vertex.

More generality and flexibility is added by extending the concept of ‘higher extensions’, that is of shifted morphisms; it is possible to replace the shift functor by any other auto-equivalence of the ambient triangulated category. There is even a version using two such functors.

The main result of this article provides a construction of derived equivalences in a setup that is very general in several respects. In the following explanation we start with a special case and then add generality step by step, finally arriving at the main result.

The setup always is a triangulated category \( T \), which is an \( R \)-category for some commutative artinian ring \( R \), with identity; so, morphism sets in \( T \) are \( R \)-modules.

1. To start with, we choose any object \( M \) in \( T \) and a triangle \( X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \rightarrow X[1] \), where \( \alpha \) and \( \beta \) are \( \text{add}(M) \)-approximations, that is universal maps from \( X \) to objects in \( \text{add}(M) \) or from \( \text{add}(M) \) to \( Y \), respectively; in particular, \( M_1 \) is in \( \text{add}(M) \). For instance, Auslander-Reiten triangles (over algebras) provide such situations. If the triangle is induced by an exact sequence in an abelian category, then the theorem implies a derived equivalence between the two endomorphism rings \( \text{End}_T(X \oplus M_1) \) and \( \text{End}_T(M_1 \oplus Y) \). This can be seen as a mutation procedure relating the two endomorphism rings. The derived equivalence has already been established in [11].

2. In the second step, recasting an idea of [12], endomorphism rings are replaced by cohomological endomorphism rings in the following sense: Higher extensions between modules \( S \) and \( T \) are shifted morphisms in the derived category, \( \text{Ext}^j(S,T) \simeq \text{Hom}(S,T[j]) \). Using Yoneda multiplication of extensions, this defines an algebra structure on the cohomological endomorphism ring, or generalised Yoneda algebra, \( \oplus_{j \in \mathbb{Z}} \text{Hom}(S,S[j]) \). When \( S \) is a complex, or any object in a triangulated category \( T \), negative degrees \( j \) may occur. The main theorem provides derived equivalences between such generalised Yoneda algebras. The construction works, however, not only for these Yoneda algebras, but also for ‘perforated’ ones in the following sense: Choose a subset \( \Phi \subset \mathbb{Z} \). Then, under some associativity constraint requiring \( \Phi \) to be ‘admissible’ (see Subsection 2.3), the space \( \oplus_{j \in \Phi} \text{Hom}(S,S[j]) \) is an associative algebra, that in general is neither a subalgebra nor a quotient algebra of the Yoneda algebra \( \oplus_{j \in \mathbb{Z}} \text{Hom}(S,S[j]) \). This algebra is called a \( \Phi \)-Yoneda algebra or a \( \Phi \)-perforated Yoneda algebra. We will use the notation \( \mathbb{E}^T_\Phi(Z) \) for the algebra \( \oplus_{j \in \Phi} \text{Hom}(Z,Z[j]) \), where \( Z \) is any object in \( T \).
The assumptions of the first step get modified by using cohomological approximations, in the degrees specified by $\Phi$, instead of approximations in degree zero only. Auslander-Reiten triangles still satisfy these properties. Adding higher extensions requires also to add an orthogonality assumption without which the result would be wrong: Assume $\text{Hom}(M,X[j]) = 0 = \text{Hom}(Y,M[j])$ for all $j \in \Phi, j \neq 0$. For the sake of exposition also assume for a moment that the above triangle $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \rightarrow X[1]$ is in a derived module category and it is induced from an exact sequence with corresponding properties. Then there are derived equivalences between $\Phi$-Yoneda algebras $\mathbb{D}^b(\mathbb{E}^\Phi_T(X \oplus M)) \simeq \mathbb{D}^b(\mathbb{E}^\Phi_T(M \oplus Y))$.

3. This result needs to be modified, if the triangle is not induced by an exact sequence any more. Then some annihilators have to be factored out of the degree zero parts of the cohomological endomorphism rings, and the derived equivalences are connecting the quotient algebras $E_{\Phi}^b(M \oplus Y)/I$ and $E_{\Phi}^b(M \oplus Y)/J$. Here, the ideals $I$ and $J$ can be described as follows: Let $\Gamma_0 = \text{End}_\mathcal{T}(M \oplus Y)$ and $e$ the idempotent element in $\Gamma_0$ corresponding to the direct summand $M$. Then $J$ is the submodule of the left $\Gamma_0$-module $\Gamma_0 e \Gamma_0$, which is maximal with respect to $eJ = 0$. Let $\Lambda_0 = \text{End}_\mathcal{T}(X \oplus M)$, and $f$ the idempotent in $\Lambda_0$ corresponding to the direct summand $M$. Then $I$ is the submodule of the right $\Lambda_0$-module $\Lambda_0 f \Lambda_0$, which is maximal with respect to $If = 0$.

Another, equivalent, description of $I$ and $J$ is that $I$ consists of all elements $(x_i)_{i \in \Phi} \in E_{\Phi}^b(M \oplus Y)$ such that $x_i = 0$ for $0 \neq i \in \Phi$ and $x_0 \alpha = 0$, and $J$ consists of all elements $(y_i)_{i \in \Phi} \in E_{\Phi}^b(M \oplus Y)$ such that $y_i = 0$ for $0 \neq i \in \Phi$ and $y_0 \beta = 0$, where $\alpha$ is the diagonal morphism $\text{diag}(\alpha, 1) : X \oplus M \rightarrow M_1 \oplus M$, and $\beta$ is the skew-diagonal morphism $\text{skewdiag}(1, \beta) : M_1 \oplus M \rightarrow M \oplus Y$.

4. The fourth level of generalisation allows to replace the shift functor by any auto-equivalence of the triangulated category $\mathcal{T}$, thus providing a new and versatile meaning of ‘higher extensions’ in terms of morphisms with one variable shifted by powers of the auto-equivalence. The additional datum $F$ gets mentioned, when necessary, in the notation as an additional superscript, as in $E_{\Phi}^b(F)(Z)$.

In this general form, the main theorem is as follows:

**Theorem 1.1.** Let $\Phi$ be an admissible subset of $\mathbb{Z}$, and let $\mathcal{T}$ be a triangulated $R$-category and $M$ an object in $\mathcal{T}$. Assume that $F$ is a triangle functor from $\mathcal{T}$ to itself, which is an auto-equivalence, that is, provided with a quasi-inverse. Suppose that

$$X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$$

is a triangle in $\mathcal{T}$ such that

1. The morphism $\alpha$ is a left $(\text{add}(M), F, \Phi)$-approximation of $X$ and $\beta$ is a right $(\text{add}(M), F, -\Phi)$-approximation of $Y$,
2. $\text{Hom}_\mathcal{T}(M, F^iX) = 0 = \text{Hom}_\mathcal{T}(F^{-i}Y, M)$ for all $0 \neq i \in \Phi$.

Then $E_{\Phi}^b_F(X \oplus M)/I$ and $E_{\Phi}^b_F(M \oplus Y)/J$ are derived equivalent, where $I$ and $J$ are the above ideals of the $\Phi$-Yoneda algebras $E_{\Phi}^b_F(X \oplus M)$ and $E_{\Phi}^b_F(M \oplus Y)$, contained in $\text{End}_\mathcal{T}(X \oplus M)$ and $\text{End}_\mathcal{T}(M \oplus Y)$, respectively.

A fifth level of generalisation, using two functors $F$ and $G$, will be discussed in the Appendix. A further generalisation of some results in this paper to $n$-angulated categories introduced in [7] will be considered in [4].

The second level of generality, where $F$ is the shift functor and both $I$ and $J$ are zero, is already widely applicable. This case happens frequently for the derived category $\mathbb{D}^b(A)$ of an $R$-algebra $A$.  

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Corollary 1.2. Let $\Phi$ be an admissible subset of $\mathbb{N}$, and let $A$ be an $R$-algebra and $M$ an $A$-module. If $0 \to X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to 0$ is an exact sequence in $A$-mod such that $\alpha$ is a left $(\text{add}(M), \Phi)$-approximation of $X$ and $\beta$ is a right $(\text{add}(M), -\Phi)$-approximation of $Y$ in $D^b(A)$, and that $\text{Ext}^i_A(M, X) = \text{Ext}^i_A(Y, M) = 0$ for all $0 \neq i \in \Phi$, then the $\Phi$-Yoneda algebras $E^{\Phi}_A(X \oplus M)$ and $E^{\Phi}_A(M \oplus Y)$ are derived equivalent.

These results partly generalise some results of [11].

The setup here, and the main result, covers, combines and extends several classical concepts:

Auslander algebras - endomorphism rings of direct sums of ‘all’ modules of an algebra of finite representation type - are the ingredients of the celebrated Auslander correspondence, characterising finite representation type via homological dimensions. Auslander algebras of derived equivalent algebras are, in general, not derived equivalent; positive results in this direction - for self-injective algebras of finite representation type - previously have been obtained in [12]. In the current approach new results can be obtained by appropriate choices of $X \oplus M$.

Another intensively studied class of algebras is that of Yoneda algebras, that is, algebras of self-extensions of a semisimple module, or more generally of any module. Apparently, the constructions in Corollary 1.2 and in [12] provide the first general class of derived equivalences for Yoneda algebras. Perforated Yoneda algebras first have been defined in [12], under the name $\Phi$-Auslander-Yoneda algebras. The approach developed there has been based on the existence of particular kinds of derived equivalences for algebras, which then have been used to construct derived equivalences for perforated Yoneda algebras.

The main novelty of the present approach is the systematic use of cohomological data, such as cohomological approximations and perforated Yoneda algebras. This relates smoothly with a wide variety of concepts, such as Auslander-Reiten sequences and triangles, dominant dimension, Calabi-Yau categories and Frobenius categories.

The article is organised as follows. In Section 2, we first fix notation, and then recall definitions and basic results on derived equivalences as well as on admissible sets and perforated Yoneda algebras. Also, we extend the notion of $D$-approximation to what we call cohomological $D$-approximation with respect to $(F, \Phi)$, where $F$ is a functor and $\Phi$ is a subset of $\mathbb{N}$. In Section 3, the main result, Theorem 1.1, is proven and various easier to access situations are described, for which the assumptions of Theorem 1.1 are satisfied. Section 4 explains how Theorem 1.1 applies to a variety of situations: derived categories of Artin algebras, Frobenius categories and Calabi-Yau categories. Also, the connection to the concept of dominant dimension is explained. In Section 5, two examples are given to illustrate the results and to show the necessity of some assumptions in Theorem 1.1. In the Appendix, a more general formulation of Theorem 1.1 is stated, which involves two functors, in order to add more flexibility with a view to potential future applications.

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2 Preliminaries

In this section, we shall recall basic definitions and facts which will be needed in the proofs later on.
2.1 Conventions

Throughout this paper, $R$ is a fixed commutative artinian ring with identity. Given an $R$-algebra $A$, by an $A$-module we mean a unitary left $A$-module; the category of all (respectively, finitely generated) $A$-modules is denoted by $A$-Mod (respectively, $A$-mod), the full subcategory of $A$-Mod consisting of all (respectively, finitely generated) projective modules is denoted by $A$-Proj (respectively, $A$-proj). There is a similar notation for right $A$-modules. The stable module category $A\text{-mod}$ of $A$ is, by definition, the quotient category of $A$-mod modulo the ideal generated by homomorphisms factorising through projective modules in $A$-proj. An equivalence between the stable module categories of two algebras is called a stable equivalence.

An $R$-algebra $A$ is called an Artin $R$-algebra if $A$ is finitely generated as an $R$-module. For an Artin $R$-algebra $A$, we denote by $D$ the usual duality on $A$-mod, and by $\nu_A$ the Nakayama functor $D\text{Hom}_A(-, AA): A$-proj $\rightarrow$ $A$-inj. For an $A$-module $M$, we denote the first syzygy of $M$ by $\Omega_A(M)$, and call $\Omega_A$ the Heller loop operator of $A$. The transpose of $M$, which is an $A^{op}$-module, is denoted by $\text{Tr}(M)$.

Let $C$ be an additive $R$-category, that is, $C$ is an additive category in which the set of morphisms between two objects in $C$ is an $R$-module, and the composition of morphisms in $C$ is $R$-bilinear. For an object $X$ in $C$, we denote by $\text{add}(X)$ the full subcategory of $C$ consisting of all direct summands of copies of $X$. An object $X$ in $C$ is called an additive generator for $C$ if $C=\text{add}(X)$. For two morphisms $f:X \rightarrow Y$ and $g:Y \rightarrow Z$ in $C$, we write $fg$ for their composition. Thus, for an $A$-module $X$, we always have a natural $A$-$\text{End}_A(X)$-bimodule structure on $X$. We shall not consider any bi-structure of categories, thus, for two functors $F:C \rightarrow D$ and $G:D \rightarrow E$, however, we write $GF$ for the composition instead of $FG$.

Given an object $M \in C$, we say that a morphism $f:X \rightarrow Y$ in $C$ factorises through $\text{add}(M)$ if there are morphisms $f_1:X \rightarrow M'$ and $f_2:M' \rightarrow Y$ in $C$ with $M' \in \text{add}(M)$ such that $f = f_1f_2$. Given a morphism $g:U \rightarrow V$ in $C$, we say that a morphism $\alpha:W \rightarrow V$ (respectively, $\beta:U \rightarrow W$) factorises through $g$ if there exists a morphism $\alpha':W \rightarrow U$ (respectively, $\beta':V \rightarrow W$) such that $\alpha = \alpha'g$ (respectively, $\beta = g\beta'$).

If $f:X \rightarrow Y$ is a map between two sets $X$ and $Y$, we denote the image of $f$ by $\text{Im}(f)$. Moreover, if $f$ is a homomorphism between two abelian groups, we denote the kernel and cokernel of $f$ by $\text{Ker}(f)$ and $\text{Coker}(f)$, respectively.

Recall that a functor $F:C \rightarrow D$ is an equivalence if there is a functor $G:D \rightarrow C$ such that $GF \simeq \text{id}_C$ and $FG \simeq \text{id}_D$. The functor $G$ is called a quasi-inverse of $F$. In this case, we write $F^{-1}$ for $G$. If $C = D$, then an equivalence $F$ is called an auto-equivalence. An auto-equivalence $F$ is called an auto-isomorphism if $F$ has a quasi-inverse $G$ such that $FG = GF = \text{id}_C$. If $F$ is a functor from $C$ to $\text{Hom}_T(X,Y[i])$. Let $\Phi$ be a subset of $Z$. An object $M$ (or a full subcategory $M$) of $T$ is called $\Phi$-self-orthogonal provided that $\text{Ext}_T^i(M,M) = 0$ (or $\text{Ext}_T^i(M,M) = 0$) for all $0 \neq i \in \Phi$, where $\text{Ext}_T^i(M,M) = 0$ means that $\text{Ext}_T^i(X,Y) = 0$ for all $X,Y \in M$. In case $\Phi = Z$, we say that $M$ is self-orthogonal. For $\Phi = \{0,1,\cdots,n\}$, we say that $M$ is $n$-self-orthogonal, which is sometimes, perhaps less suggestively, referred to as $n$-rigid.

Replacing the shift functor by a triangle auto-equivalence $F$, one may also define the notion of $(F,\Phi)$-self-orthogonality, but we refrain from introducing this notion here.

2.2 Derived equivalences

Let $C$ be an additive $R$-category.

By a complex $X^*$ over $C$ we mean a sequence of morphisms $d_X^i$ between objects $X^i$ in $C$: $\cdots \rightarrow X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} X^{i+2} \rightarrow \cdots$, such that $d_X^id_X^{i+1} = 0$ for all $i \in \mathbb{Z}$; we write $X^* = (X^i, d_X^i)$. For a complex
$X^\bullet$, the **brutal truncation** $\sigma_{<i} X^\bullet$ of $X^\bullet$ is a quotient complex of $X^\bullet$ such that $(\sigma_{<i} X^\bullet)^k$ is $X^k$ for all $k < i$ and zero otherwise. Similarly, we define $\sigma_{\geq i} X^\bullet$. For a fixed $n \in \mathbb{Z}$, we denote by $X^\bullet[n]$ the complex obtained from $X^\bullet$ by shifting degree by $n$, that is, $(X^\bullet[n])^n = X^n$.

The category of all complexes over $\mathcal{C}$ with chain maps is denoted by $\mathcal{C}(\mathcal{C})$. The homotopy category of complexes over $\mathcal{C}$ is denoted by $\mathcal{K}(\mathcal{C})$. When $\mathcal{C}$ is an abelian category, the derived category of complexes over $\mathcal{C}$ is denoted by $\mathcal{D}(\mathcal{C})$. The full subcategories of $\mathcal{K}(\mathcal{C})$ and $\mathcal{D}(\mathcal{C})$ consisting of bounded complexes over $\mathcal{C}$ are denoted by $\mathcal{K}^b(\mathcal{C})$ and $\mathcal{D}^b(\mathcal{C})$, respectively. As usual, for an algebra $A$, we simply write $\mathcal{C}(A)$ for $\mathcal{C}(A,\text{mod})$, $\mathcal{K}(A)$ for $\mathcal{K}(A,\text{mod})$ and $\mathcal{K}^b(A)$ for $\mathcal{K}^b(A,\text{mod})$. Similarly, we write $\mathcal{D}(A)$ and $\mathcal{D}^b(A)$ for $\mathcal{D}(A,\text{mod})$ and $\mathcal{D}^b(A,\text{mod})$, respectively.

For an $R$-algebra $A$, the categories $\mathcal{K}(A)$ and $\mathcal{D}(A)$ are triangulated $R$-categories. For basic results on triangulated categories, we refer the reader to [9] and [19].

The following result, due to Rickard (see [21, Theorem 6.4]) by a direct approach, and to Keller by working in the more general setup of differential graded algebras, is fundamental in the investigation of derived equivalences.

\textbf{Theorem 2.1.} [21] Let $\Lambda$ and $\Gamma$ be two rings. The following conditions are equivalent:

(a) $\mathcal{K}^-(\Lambda,\text{proj})$ and $\mathcal{K}^-(\Gamma,\text{proj})$ are equivalent as triangulated categories;

(b) $\mathcal{D}^b(\Lambda,\text{mod})$ and $\mathcal{D}^b(\Gamma,\text{mod})$ are equivalent as triangulated categories;

(c) $\mathcal{K}^b(\Lambda,\text{proj})$ and $\mathcal{K}^b(\Gamma,\text{proj})$ are equivalent as triangulated categories;

(d) $\mathcal{K}^b(\Lambda,\text{proj})$ and $\mathcal{K}^b(\Gamma,\text{proj})$ are equivalent as triangulated categories;

(e) $\Gamma$ is isomorphic to $\text{End}_{\mathcal{K}^b(\Lambda,\text{proj})}(T^\bullet)$, where $T^\bullet$ is a complex in $\mathcal{K}^b(\Lambda,\text{proj})$ satisfying:

(1) $T^\bullet$ is self-orthogonal, that is, $\text{Hom}_{\mathcal{K}^b(\Lambda,\text{proj})}(T^\bullet,T^\bullet[i]) = 0$ for all $i \neq 0$.

(2) $\text{add}(T^\bullet)$ generates $\mathcal{K}^b(\Lambda,\text{proj})$ as a triangulated category.

Two rings $\Lambda$ and $\Gamma$ are called derived equivalent if the above conditions (a)-(e) are satisfied. A complex $T^\bullet$ in $\mathcal{K}^b(\Lambda,\text{proj})$ as above is called a tilting complex over $\Lambda$.

For Artin algebras, the above equivalent conditions can be reformulated in terms of finitely generated modules: Two Artin $R$-algebras $A$ and $B$ are said to be derived equivalent if their derived categories $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories. In this case, there is a tilting complex $T^\bullet$ in $\mathcal{K}^b(A,\text{proj})$ such that $B \simeq \text{End}_{\mathcal{K}^b(A)}(T^\bullet)$.

2.3 Admissible subsets and $\Phi$-Oneda algebras

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of natural numbers, and let $\mathbb{Z}$ be the set of all integers. For a natural number $n$ or infinity, let $\mathbb{N}_n := \{i \in \mathbb{N} \mid 0 \leq i < n+1\}$.

Recall from [12] that a subset $\Phi$ of $\mathbb{Z}$ containing 0 is called an admissible subset of $\mathbb{Z}$ if the following condition is satisfied:

\[\text{If } i, j \text{ and } k \text{ are in } \Phi \text{ such that } i + j + k \in \Phi, \text{ then } i + j \in \Phi \text{ if and only if } j + k \in \Phi.\]

Clearly, if $\Phi$ is an admissible subset of $\mathbb{Z}$, then so is $-\Phi := \{-x \mid x \in \Phi\}$. Any subset $\{0, i, j\}$ of $\mathbb{N}$ is an admissible subset of $\mathbb{Z}$. Moreover, for any subset $\Phi$ of $\mathbb{N}$ containing zero and for any positive integer $m \geq 3$, the set $\{am \mid a \in \Phi\}$ is admissible in $\mathbb{Z}$ (for more examples, see [12]). Nevertheless, not every subset of $\mathbb{N}$ containing zero is admissible, for instance, $\{0, 1, 2, 4\}$ is not admissible. In fact, this is the ‘smallest’ nonadmissible subset of $\mathbb{N}$.

Admissible sets were used to define $\Phi$-Oneda algebras in [12], under the name of ‘$\Phi$-Auslander-Yoneda algebras’. The formulation there works more generally for monoid graded algebras. For our purpose in this paper, we restrict to the case of an object in a triangulated category.

Let $\Phi$ be an admissible subset of $\mathbb{Z}$, and let $\mathcal{T}$ be a triangulated $R$-category with a shift functor $[1]$. Suppose that $F$ is a triangle functor from $\mathcal{T}$ to $\mathcal{T}$. Recall that we put $F^i = 0$ for $i < 0$ if $F^{-1}$ does not exist.
Let $E^F_{\mathcal{T}}(-,-)$ be the bi-functor

$$
\bigoplus_{i \in \Phi} \text{Hom}_{\mathcal{T}}(-,F^i-): \mathcal{T} \times \mathcal{T} \to \text{R-Mod},
$$

$$(X,Y) \mapsto E^F_{\mathcal{T}}(X,Y) := \bigoplus_{i \in \Phi} \text{Hom}_{\mathcal{T}}(X,F^iY),$$

$X \xrightarrow{f} X' \mapsto \bigoplus_{i \in \Phi} \text{Hom}_{\mathcal{T}}(f,F^iY), \quad Y \xrightarrow{g} Y' \mapsto \bigoplus_{i \in \Phi} \text{Hom}_{\mathcal{T}}(X,F^ig).$$

Suppose that $X,Y$ and $Z$ are objects in $\mathcal{T}$. Let $f = (f_i)_{i \in \Phi} \in E^F_{\mathcal{T}}(X,Y)$ and $g = (g_i)_{i \in \Phi} \in E^F_{\mathcal{T}}(Y,Z)$. We define a composition as follows:

$$E^F_{\mathcal{T}}(X,Y) \times E^F_{\mathcal{T}}(Y,Z) \to E^F_{\mathcal{T}}(X,Z),$$

$$(f,g) \mapsto fg := \left( \sum_{u \in \Phi} f_u * g_v \right)_{i \in \Phi},$$

where $f_u * g_v := f_u(F(g)v)\chi(u,v)$ with $\chi(u,v)$ being a natural transformation from $F^uF^v$ to $F^{u+v}$ defined as follows.

If $F^{-1}$ does not exist, then $\chi(u,v) = 0$ if $u$ or $v$ is negative, and $\chi(u,v) = \text{id}_{F^u \circ F^v}$ otherwise.

If $F^{-1}$ exists, then $(F,F^{-1})$ is an adjoint pair. Let $\varepsilon : id_{\mathcal{T}} \to FF^{-1}$ be the unit and let $\eta : F^{-1}F \to id_{\mathcal{T}}$ be the counit. The natural transformation $\chi(u,v)$ is defined to be $\text{id}_{F^u \circ F^v}$ if $uv \geq 0$. If $u > 0$ and $v < 0$, then $\chi(u,v)$ is defined as the composite of the sequence of natural transformations

$$F^u F^v \xrightarrow{\eta_{u+1}} F^{u-1} F^{v+1} \xrightarrow{F^{-2} \eta_{u+1}} F^{u-2} F^{v+2} \to \cdots \xrightarrow{F^{-i} F^{v+i}} F^{u+v}.$$ 

If $u < 0$ and $v > 0$, then $\chi(u,v)$ is defined as the composite of the following natural transformations

$$F^u F^v \xrightarrow{- \eta_{-i}} \overline{F}^u F^v \xrightarrow{F^{-2} \eta_{-i}} \overline{F}^{u+2} F^{v-2} \to \cdots \xrightarrow{F^{-i} F^{v-i}} F^{u+v}.$$ 

In this setting, the above-defined composition is associative. To prove this, one needs to check that the multiplication $f_u * g_v$ is associative. This follows if the following diagram is commutative

$$F^i F^j F^k \xrightarrow{\eta_{i+j+k}} F^i F^{j+k} \xrightarrow{\chi(i,j,k)} F^{i+j+k} \xrightarrow{\chi(i+j,k)} F^{i+j+k}.$$

for all integers $i,j,k \in \Phi$. However, using the fact that $F(\varepsilon)\eta_F = \text{id}_F$ and $\eta_{F^{-1}} F^{-1}(\varepsilon) = \text{id}_{F^{-1}}$, one can get the above commutative diagram by drawing a big commutative diagram with the above two sequences of natural transformations. Here we leave it to the reader. Note that if $FF^{-1} = \text{id}_{\mathcal{T}} = F^{-1}F$, then $\chi(u,v)$ is an identity for all $u,v \in \mathbb{Z}$, and therefore will not appear in the definition of the multiplication.

Thus $E^F_{\mathcal{T}}(X,X)$ is an $R$-algebra. It is called the $\Phi$-Yoneda algebra or, when $\Phi$ is fixed, the perforated Yoneda algebra of $X$ with respect to $F$. Then $E^F_{\mathcal{T}}(X,Y)$ is a left $E^F_{\mathcal{T}}(X,X)$-module. When $\Phi = \mathbb{N}$, the algebra $E^F_{\mathcal{T}}(X,X)$ is the orbit algebra of $X$ under $F$ (see [2]).

For convenience we write $E^F_{\mathcal{T}}(X,Y)$ for $E^F_{\mathcal{T}}(X,Y)$, $E^F_{\mathcal{T}}(X)$ for $E^F_{\mathcal{T}}(X,X)$, and $E^F_{\mathcal{T}}(Y)$ for $E^F_{\mathcal{T}}(Y,Y)$. 


When $F$ coincides with the shift functor, we omit the upper index $F$, and call $E^*_F(X)$ the $\Phi$-Yoneda algebra of $X$, without referring to the shift functor. This is the algebra introduced in [12] and there called an Auslander-Yoneda algebra.

The following lemma is essentially taken from [12, Lemma 3.5], where a variation of it appears. The proof given there carries over to the present situation.

**Lemma 2.2.** Let $\mathcal{T}$ be a triangulated $R$-category with a triangle endo-functor $F$, and let $U$ be an object in $\mathcal{T}$. Suppose that $U_1$, $U_2$ and $U_3$ are in $\text{add}(U)$, and that $\Phi$ is an admissible subset of $\mathbb{Z}$. Then,

1. there is a natural isomorphism
   $$\mu: E^F_{\mathcal{T}}(U_1, U_2) \rightarrow \text{Hom}_{E^F_{\mathcal{T}}(U)}(E^F_{\mathcal{T}}(U, U_1), E^F_{\mathcal{T}}(U, U_2)),$$
   which sends $x \in E^F_{\mathcal{T}}(U_1, U_2)$ to the morphism $a \mapsto ax$ for $a \in E^F_{\mathcal{T}}(U, U_1)$. Moreover, if $x \in E^F_{\mathcal{T}}(U_1, U_2)$ and $y \in E^F_{\mathcal{T}}(U_2, U_3)$, then $\mu(xy) = \mu(x)\mu(y)$.

2. The functor $E^F_{\mathcal{T}}(U, -) : \text{add}(U) \rightarrow E^F_{\mathcal{T}}(U)-\text{proj}$ is faithful.

3. If $\text{Hom}_{\mathcal{T}}(U_1, F^iU_2) = 0$ for all $i \in \Phi \setminus \{0\}$, then the functor $E^F_{\mathcal{T}}(U, -)$ induces an isomorphism of $R$-modules:
   $$E^F_{\mathcal{T}}(U, -) : \text{Hom}_{\mathcal{T}}(U_1, U_2) \rightarrow \text{Hom}_{E^F_{\mathcal{T}}(U)}(E^F_{\mathcal{T}}(U, U_1), E^F_{\mathcal{T}}(U, U_2)).$$

The properties described in Lemma 2.2 will be frequently used in the proofs below.

The class of $\Phi$-Yoneda algebras with respect to a functor includes a large class of algebras, for example the following:

(a) The endomorphism algebra of a module, in particular, the Auslander algebras of representation-finite algebras. Here we choose $\Phi = \{0\}$.

(b) The generalised Yoneda algebra of a module if we take $\Phi = \mathbb{N}$. This includes the preprojective algebras (see [2]) and the Hochschild cohomology rings of given algebras. Choosing $\Phi = 2\mathbb{N}$, we get for instance the even Hochschild cohomology rings of algebras.

(c) Certain trivial extensions: For an Artin algebra $A$ and an $A$-module $M$ we choose $\Phi = \{0, i\}$ for $i \geq 1$ an arbitrary natural number. Then $E^\Phi_{A}(M)$ is the trivial extension of $\text{End}_A(M)$ by the bimodule $\text{Ext}^i_A(M, M)$. Such rings appear naturally in the (bounded) derived category $\mathcal{D}^b(\mathbf{X})$ of coherent sheaves of a smooth projective variety $\mathbf{X}$ over $\mathbb{C}$. Indeed, if $X$ is a $d$-spherical object in $\mathcal{D}^b(\mathbf{X})$, then its cohomological ring $\text{End}_{\mathcal{D}^b(\mathbf{X})}(X)$ is $E^{\{0,d\}}_{\mathcal{D}^b(\mathbf{X})}(X)$, this is a graded ring isomorphic to $\mathbb{C}[t]/(t^d)$ with $t$ of degree $d$. For further information on spherical objects, we refer the reader to [23, Section 3c].

In general, if $\Phi = \{0, a_1, \ldots, a_n\} \subseteq \mathbb{N}$ such that $a_i > 2a_{i-1}$ for $i = 2, \cdots, n$, then $E^\Phi_{A}(X)$ is the trivial extension of $\text{End}_A(X)$ by the bimodule $\bigoplus_{0 \neq i \in \Phi} \text{Ext}^i_A(X, X)$. Note that $\Phi = \{0\} \cup \{2n + 1 \mid n \in \mathbb{N} \}$ is admissible. In this case, we also get a trivial extension.

(d) The polynomial ring $R[t]$; If we take $\Phi = m\mathbb{N}$ for $m \geq 1$, then the perforated Yoneda algebra $E^\Phi_{R[t]/(t^m)}(R)$ is isomorphic to $R[t^m]$ with $t$ a variable. If $\Phi = \{0, 1, \cdots, n\}$, then $E^\Phi_{R[t]/(t^m)}(R) \simeq R[t]/(t^n)$.

### 2.4 $\mathcal{D}$-split sequences and cohomological $\mathcal{D}$-approximations

$\mathcal{D}$-split sequences have been defined in [11] in the context of constructing derived equivalences between certain endomorphism algebras. Let us recall the definition and a result in [11].

Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ a full subcategory of $\mathcal{C}$. A sequence

$$X \xrightarrow{f} M \xrightarrow{g} Y$$

is said to be a $\mathcal{D}$-split sequence if there exists an $\mathcal{D}$-approximation

$$X \rightarrow X' \rightarrow Y.$$
in \( C \) is called an \( \mathcal{D} \)-split sequence if

1. \( M \in \mathcal{D} \),
2. \( f \) is a left \( \mathcal{D} \)-approximation of \( X \), and \( g \) is a right \( \mathcal{D} \)-approximation of \( Y \), and
3. \( f \) is a kernel of \( g \), and \( g \) is a cokernel of \( f \).

Typical examples of \( \mathcal{D} \)-split sequences are Auslander-Reiten sequences. Every \( \mathcal{D} \)-split sequence provides a derived equivalence (see [11, Theorem 1.1]). Here are some details, for later reference.

**Theorem 2.3.** [11] Let \( C \) be an additive category, and \( M \) an object in \( C \). Suppose that

\[
X \longrightarrow M' \longrightarrow Y
\]

is an add\((M)\)-split sequence in \( C \). Then the endomorphism ring \( \text{End}_C(M \oplus X) \) of \( M \oplus X \) is derived-equivalent to the endomorphism ring \( \text{End}_C(M \oplus Y) \) of \( M \oplus Y \) via a tilting module of projective dimension at most 1.

Now, the question arises whether Theorem 2.3 can be extended to \( \Phi \)-Yoneda algebras. The second example in the final section demonstrates that this is no longer true if we just replace the endomorphism algebras in Theorem 2.3 by \( \Phi \)-Yoneda algebras. Nevertheless, we shall show that under certain orthogonality conditions, there still is a positive answer. This will be discussed in detail in the next section.

The condition (3) of a \( \mathcal{D} \)-split sequence are substitutes in this general setup for requiring the short exact sequence to be exact. Since triangles in triangulated categories are replacements of short exact sequences, we may reformulate the notion of \( \mathcal{D} \)-split sequences in the following sense for triangulated categories.

Let \( \mathcal{T} \) be a triangulated category with a shift functor \([1]\), and let \( \mathcal{D} \) be a full additive subcategory of \( \mathcal{T} \). A triangle

\[
X \xrightarrow{\alpha} M' \xrightarrow{\beta} Y \longrightarrow X[1]
\]

in \( \mathcal{T} \) is called a \( \mathcal{D} \)-split triangle if \( M' \in \mathcal{D} \), the map \( \alpha \) is a left \( \mathcal{D} \)-approximation of \( X \) and the map \( \beta \) is a right \( \mathcal{D} \)-approximation of \( Y \).

Thus, for an Artin \( R \)-algebra \( A \), every \( \mathcal{D} \)-split sequence in \( A \text{-mod} \) extends to a \( \mathcal{D} \)-split triangle in \( \mathcal{D}^b(A) \).

Next, we introduce the left and right cohomological \( \mathcal{D} \)-approximations with respect to \((F, \Phi)\), which generalise the notions of left and right \( \mathcal{D} \)-approximations, respectively.

Suppose that \( C \) is a category with an endo-functor \( F : C \to C \). Let \( \mathcal{D} \) be a full subcategory of \( C \), and let \( \Phi \) be a non-empty subset of \( \mathbb{N} \). If \( F \) has an inverse, then \( \Phi \) may be chosen to be a subset of \( \mathbb{Z} \). Suppose that \( X \) is an object of \( C \). A morphism \( f : X \to D \) in \( C \) is called a left cohomological \( \mathcal{D} \)-approximation of \( X \) with respect to \((F, \Phi)\) (or shortly, a left \((\mathcal{D}, F, \Phi)\)-approximation of \( X \)) if \( D \in \mathcal{D} \), and for any morphism \( g : X \to F^i(D') \) with \( D' \in \mathcal{D} \) and \( i \in \Phi \), there is a morphism \( g' : D \to F^i(D') \) such that \( g = fg' \). Here \( F^0 = \text{id}_C \). Similarly, we have the notion of a right \((\mathcal{D}, F, \Phi)\)-approximation of \( X \) in \( \mathcal{T} \), that is, a morphism \( f : D \to X \) with \( D \) in \( \mathcal{D} \) is called a right \((\mathcal{D}, F, \Phi)\)-approximation of \( X \) if, for any \( i \in \Phi \) and any morphism \( g : F^iD' \to X \) with \( D' \in \mathcal{D} \), there is a morphism \( g' : F^iD' \to D \) such that \( g = g'f \).

Note that if \( F = \text{id}_C \) and \( \Phi = \{0\} \), then we get the original notion of approximations in the sense of Auslander and Smalø. (In ring theory, such approximations are called pre-envelope and precover, respectively). Moreover, if \( 0 \in \Phi \), then every left \((\mathcal{D}, F, \Phi)\)-approximation of \( X \) is also a left \( \mathcal{D} \)-approximation of \( X \), and every right \((\mathcal{D}, F, \Phi)\)-approximation of \( X \) is also a right \( \mathcal{D} \)-approximation of \( X \).
If \( F = [1] \) and \( T = \mathcal{O}^N(A) \) for an Artin algebra \( A \), then \( \text{Hom}_T(X,F^iY) \simeq \text{Ext}^i_A(X,Y) \) for all \( X,Y \in A\text{-mod} \) and all \( i \geq 0 \). For this reason, a \((\mathcal{D},F,\Phi)\)-approximation has been called a cohomological approximation.

In this paper, we are mainly interested in the case where \( C \) is a triangulated \( R\)-category \( \mathcal{T} \) with a triangle endo-functor \( F \), and \( \mathcal{D} \) is a full subcategory of \( \mathcal{T} \). Thus, a morphism \( f : X \to D \) with \( D \in \mathcal{D} \) and \( X \in \mathcal{T} \) is a left \((\mathcal{D},F,\Phi)\)-approximation of \( X \) if and only if the canonical map \( E_{\mathcal{T}}^F(\Phi,f,\Phi) : E_{\mathcal{T}}^F(D,D) \to E_{\mathcal{T}}^F(X,X) \), defined by \((x_i)_{i \in \Phi} \mapsto (f x_i)_{i \in \Phi}\), is surjective for all \( D' \in \mathcal{D} \). Similarly, a morphism \( g : D \to X \) with \( D \in \mathcal{D} \) and \( X \in \mathcal{T} \) is a right \((\mathcal{D},F,\Phi)\)-approximation of \( X \) if and only if the canonical map \( \text{Hom}_\mathcal{T}(F^iD',g) : \text{Hom}_\mathcal{T}(F^iD',D) \to \text{Hom}_\mathcal{T}(F^iD',X) \) is surjective for every \( D' \in \mathcal{D} \) and \( j \in \Phi \). If, moreover, \( F \) is an invertible triangle functor, then a morphism \( g : D \to X \) with \( D \in \mathcal{D} \) and \( X \in \mathcal{T} \) is a right \((\mathcal{D},F,\Phi)\)-approximation of \( X \) if and only if the canonical map \( E_{\mathcal{T}}^{F^{-1}\Phi}(D',g) : E_{\mathcal{T}}^{F^{-1}\Phi}(D',D) \to E_{\mathcal{T}}^{F^{-1}\Phi}(D',X) \) is surjective for all \( D' \in \mathcal{D} \). Note that here we need the minus sign for \( \Phi \) and that \( F^{-1} \) exists.

If \( F \) coincides with the shift functor \([1]\), we simply speak of \((\mathcal{D},\Phi)\)-approximations, without mentioning \( F \).

Note that if \( F \) contains zero and if \( \text{Hom}_\mathcal{T}(F^iM,0) = 0 \) for all \( i \neq 0 \in \Phi \) and \( D' \in \mathcal{D} \), then \( f \) is a left \((\mathcal{D},F,\Phi)\)-approximation of \( X \) if and only if \( f \) is a left \( \mathcal{D} \)-approximation of \( X \). A dual statement is also true for a right \((\mathcal{D},F,\Phi)\)-approximation of \( X \).

Here is a source of examples of \((\mathcal{D},\Phi)\)-approximations. Suppose that \( \mathcal{T} = \mathcal{O}^N(A) \) for an Artin \( R\)-algebra and that \( \Phi \) is a subset of \( \mathbb{Z} \). Let \( X \xrightarrow{\alpha} M \xrightarrow{\beta} Y \to X[1] \) be an Auslander-Reiten triangle in \( \mathcal{T} \). If neither \( X \) nor \( Y \) belongs to \( \text{add}(M[\{i\}]) \) for every \( i \neq 0 \in \Phi \), then \( \alpha \) is a left \((\text{add}(M),\Phi)\)-approximation of \( X \), and \( \beta \) is a right \((\text{add}(M),\Phi)\)-approximation of \( Y \).

Finally, we note that the difference of a left \((\mathcal{D},F,\Phi)\)-approximation of \( X \) from a left \((\bigcup_{i \in \Phi}F^i\mathcal{D})\)-approximation of \( X \) in the sense of Auslander and Smalø, where \( \bigcup_{i \in \Phi}F^i\mathcal{D} \) is the full subcategory of \( \mathcal{T} \) with all objects in \( F^i\mathcal{D} \) for all \( i \in \Phi \). Suppose \( 0 \in \Phi \). Then a \((\mathcal{D},F,\Phi)\)-approximation is a \((\bigcup_{i \in \Phi}F^i\mathcal{D})\)-approximation, but the converse is not true in general. If \( 0 \notin \Phi \), then the two concepts are independent.

So, roughly speaking, a cohomological \( \mathcal{D} \)-approximation with respect to \((F,\Phi)\) emphasises not only the factorisations but also that the object belongs to the given subcategory \( \mathcal{D} \) (and not to \( F^i\mathcal{D} \) for \( i \neq 0 \in \Phi \)).

3 Derived equivalences for \( \Phi \)-Yoneda algebras

In this section, we shall prove Theorem 1.1 and derive some consequences and some simplifications in special cases.

Suppose that \( \mathcal{T} \) is a triangulated \( R\)-category with a shift functor \([1]\), and \( M \) is an object in \( \mathcal{T} \). Suppose that \( F \) is a triangle auto-equivalence of \( \mathcal{T} \), which may be different from the shift functor.

For a subset \( \Phi \) of \( \mathbb{Z} \), we define \(-\Phi := \{-x \mid x \in \Phi\}\), and

\[
\mathcal{B}_T^{F,\Phi}(M) = \{X \in \mathcal{T} \mid \text{Hom}_\mathcal{T}(X,F^iM) = 0 \quad \text{for all} \quad i \in \Phi \setminus \{0\}\},
\]

\[
\mathcal{B}_T^{F,\Phi}(M) = \{Y \in \mathcal{T} \mid \text{Hom}_\mathcal{T}(M,F^iY) = 0 \quad \text{for all} \quad i \in \Phi \setminus \{0\}\}.
\]

Let \( n \) be a positive integer. For simplicity, we write \( \mathcal{B}_T^{F,n}(M) \) for \( \mathcal{B}_T^{F,0,1,2,\ldots,n}(M) \), and \( \mathcal{B}_T^{F,\infty}(M) \) for \( \mathcal{B}_T^{F,n}(M) \) if \( \mathcal{T} \) is clear in the context. Similarly, the notations \( \mathcal{B}_T^{F,n}(M) \) and \( \mathcal{B}_T^{F,\infty}(M) \) are defined.

As usual, \( F \) is omitted in notation when it coincides with the shift functor.

Given a triangle \( X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1] \) in \( \mathcal{T} \) with \( M_1 \in \text{add}(M) \), we define

\[
\bar{w} = (w,0) : Y \longrightarrow (X \oplus M)[1], \quad \bar{w} = (0,w)^T : M \oplus Y \longrightarrow X[1],
\]

where \((0,w)^T\) stands for the transpose of the matrix \((0,w)\), and
The main result of this paper is the following theorem which is a reformulation of Theorem 1.1.

**Theorem 3.1.** Let $\Phi$ be an admissible subset of $\mathbb{Z}$, let $T$ be a triangulated $R$-category with a triangle auto-equivalence $F$, and let $M$ be an object in $T$. Suppose that

$$X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$$

is a triangle in $T$ such that the morphism $\alpha$ is a left $(\text{add}(M), F, \Phi)$-approximation of $X$, that the morphism $\beta$ is a right $(\text{add}(M), F, -\Phi)$-approximation of $Y$ and that $X \in \mathcal{D}^{-F, \Phi}(M)$ and $Y \in \mathcal{D}^{-F, \Phi}(M)$. Then the algebras $E^{F, \Phi}_T(X \oplus M)/I$ and $E^{F, \Phi}_T(M \oplus Y)/J$ are derived equivalent.

**Proof.** Let $V = X \oplus M$ and $W = M \oplus Y$. Set

$$\tilde{\alpha} := (\alpha, 0) : X \rightarrow M_1 \oplus M, \quad \tilde{\beta} := \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} : M_1 \oplus M \rightarrow M \oplus Y, \quad \tilde{w} := \begin{pmatrix} 0 \\ w \end{pmatrix} : M \oplus Y \rightarrow X[1];$$

$$\tilde{\alpha} := \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} : X \oplus M \rightarrow M_1 \oplus M, \quad \tilde{\beta} := \begin{pmatrix} \beta \\ 0 \end{pmatrix} : M_1 \oplus M \rightarrow Y, \quad \tilde{w} := (w, 0) : Y \rightarrow (X \oplus M)[1].$$

Then there are two triangles in $T$:

$$X \xrightarrow{\tilde{\alpha}} M_1 \oplus M \xrightarrow{\tilde{\beta}} W \xrightarrow{\tilde{w}} X[1],$$

$$Y[-1] \xrightarrow{-\tilde{w}[−1]} V \xrightarrow{\tilde{\alpha}} M_1 \oplus M \xrightarrow{\tilde{\beta}} Y.$$

Since $F$ is a triangle functor, there is a natural isomorphism $\delta : F[1] \rightarrow [1]F$. That is, for any object $X$ in $T$, there is an isomorphism $\delta_X : F(X[1]) \rightarrow (FX)[1]$, which is natural in $X$. The isomorphism $F^i(X[j]) \rightarrow (F^iX)[j]$ is denoted by $\delta(F, i, X, j)$. In part II of this article, there will be further discussion of these natural transformations.

**Lemma 3.2.** (1) For any morphism $y_i : V \rightarrow F^iV$ with $i \in \Phi$, there is a morphism $t_i : Y[−1] \rightarrow (F^iY)[−1]$ such that $(\tilde{w}[−1])y_i = t_i\delta(F, i, Y, −1)^{−1}(F^i(\tilde{w}[−1])).$

(2) For any morphism $x_i : W \rightarrow F^iW$ with $i \in \Phi$, there is a morphism $t_i : X[1] \rightarrow (F^iX)[1]$ such that $x_i(F^i\tilde{w})\delta(F, i, X, 1) = \tilde{w}_i$t.

**Proof.** (1) Note that $\tilde{\alpha}$ is a left $(\text{add}(M), F, \Phi)$-approximation of $V$. Thus, given $y_i : V \rightarrow F^iV$, there is a morphism $z_i : M_1 \oplus M \rightarrow F^i(M_1 \oplus M)$ such that $\tilde{\alpha}z_i = y_i(F^i(\tilde{\alpha}))$. Since $F$ is a triangle functor, the second triangle implies that there is a triangle (see [9, p.4])

$$(F^iY)[−1] \xrightarrow{\delta(F, i, Y, −1)^{−1}(−(F^i(\tilde{w}[−1])))} F^iV \xrightarrow{F^i\tilde{\alpha}} F^i(M_1 \oplus M) \xrightarrow{F^i\tilde{\beta}} F^iY.$$

Thus there is a morphism $t_i : Y[−1] \rightarrow (F^iY)[−1]$ such that $(\tilde{w}[−1])y_i = t_i\delta(F, i, Y, −1)^{−1}(F^i(\tilde{w}[−1])).$

(2) The proof of (2) is similar to that of (1), using the following triangle

$$F^iX \xrightarrow{F^i\tilde{\alpha}} F^i(M_1 \oplus M) \xrightarrow{F^i\tilde{\beta}} F^iW \xrightarrow{(F^i\tilde{w})\delta(F, i, X, 1)} (F^iX)[1].$$

Now we prove that the quotient rings in Theorem 3.1 are well defined.

\[
I := \{ x = (x_i) \in E^{F, \Phi}_T(X \oplus M) \mid x_i = 0 \text{ for } 0 \neq i \in \Phi, x_0 \text{ factorises through } \text{add}(M) \text{ and } \hat{w}[-1] \},
\]

$$J := \{ y = (y_i) \in E^{F, \Phi}_T(M \oplus Y) \mid y_i = 0 \text{ for } 0 \neq i \in \Phi, y_0 \text{ factorises through } \text{add}(M) \text{ and } \hat{w} \}. $$

The sets $I$ and $J$ are indeed independent of $F$ and $\Phi \setminus \{0\}$, and contained in $\text{End}_T(X \oplus M)$ and $\text{End}_T(M \oplus Y)$, respectively.
Lemma 3.3. The I and J appearing in Theorem 3.1 are ideals of $E^{F \Phi}_T(V)$ and $E^{F \Phi}_T(W)$, respectively.

Proof. We shall only prove that $I$ is an ideal in $E^{F \Phi}_T(V)$. The proof for $J$ can be carried out analogously.

The set $I$ is closed under addition in $E^{F \Phi}_T(V)$. To show that $I$ is a two-sided ideal in $E^{F \Phi}_T(V)$, we pick an $x = (x_i)_{i \in \Phi} \in I$ and a $y = (y_i)_{i \in \Phi} \in E^{F \Phi}_T(V)$, and calculate the products $xy$ and $yx$ in $E^{F \Phi}_T(V)$. We write $x_0 = uv$ for $u \in V \to M'$ and $v : M' \to V$, where $M'$ is an object in add($M$), and $x_0 = s(\tilde{w}[-1])$ for a morphism $s : V \to Y[-1]$. Note that $xy = (x_0y_i)_{i \in \Phi}$ and $yx = (y_iF^x_0)_{i \in \Phi}$ since $x_0 = 0$ for $0 \neq i \in \Phi$.

We first show that $I$ is a right ideal.

(1) Let $i = 0$. The map $x_0y_0$ factorises through an object in add($M$). Since $x_0$ factorises through $\tilde{w}[-1]$, it follows from Lemma 3.2 (1) that $x_0y_0$ factorises also through $\tilde{w}[-1]$.

(2) Let $0 \neq i \in \Phi$. In this case, Hom$_T(M, F^iX) = 0$ by the assumption $X \in \mathscr{F}^F \Phi(M)$. Let $p_X$ and $p_M$ be the projections of $V$ onto $X$ and $M$, respectively. Then the composition $v y_i F^i p_X : M' \to V \to F^i V \to F^i X$ belongs to Hom$_T(M', F^iX) = 0$. Thus $x_0y_i F^i p_X = uv y_i F^i p_X = 0$. By Lemma 3.2 (1), there is a morphism $\eta : Y[-1] \to F^i Y[-1]$ such that $(\tilde{w}[-1])y_i = \eta(p, i, Y, -1)^{-1} F^i (\tilde{w}[-1])$. Hence

$$x_0y_i (F^i p_M) = s(\tilde{w}[-1])y_i (F^i p_M) = st \delta(F, i, Y, -1)^{-1} F^i (\tilde{w}[-1]) (F^i p_M)$$

$$= st \delta(F, i, Y, -1)^{-1} F^i ((w[-1], 0)(i_\mu)) = 0.$$

Altogether, $x_0y_i = x_0y_i (F^i p_M) = 0$ for $0 \neq i \in \Phi$, hence $xy \in I$, and $I$ is a right ideal in $E^{F \Phi}_T(V)$.

Next, we show that $I$ is a left ideal, that is, we check $(y_i F^x_0)_{i \in \Phi} \in I$.

(3) The map $y_0 x_0$ factorises through an object in add($M$) and through $\tilde{w}[-1]$.

(4) Let $0 \neq i \in \Phi$. Note that $\tilde{\alpha} : V \to M_1 \oplus M$ is a left (add($M$), $F$, $\Phi$)-approximation of $V$. Thus there is a morphism $h_i : M_1 \oplus M \to F^i (M')$ such that $y_i (F^i u) = \tilde{\alpha} h_i$. By assumption, Hom$_T(M, F^iX) = 0$. This implies that $h_i (F^i y_i) (F^i p_X) = 0$, and therefore $y_i (F^i x_0) (F^i p_X) = 0$. Since $(F^i \tilde{w}[-1]) (F^i p_M) = 0$, we get $y_i (F^i x_0) (F^i p_M) = 0$. Thus $y_i F^x_0 = 0$ for $0 \neq i \in \Phi$.

Hence $yx \in I$, and $I$ is a left ideal in $E^{F \Phi}_T(V)$. Thus $I$ is an ideal in $E^{F \Phi}_T(V)$.

We know that $E^{F \Phi}_T(V, Z)$ is a $E^{F \Phi}_T(V)$-module for any object $Z$ in $T$. The next lemma shows that the ideal $I$ of $E^{F \Phi}_T(V)$ may annihilate some modules of this form.

Lemma 3.4. Keep the notations as above. Then

(1) $I \cdot E^{F \Phi}_T(V, M) = 0$.

(2) $I \cdot E^{F \Phi}_T(V, X) = \{(x_i)_{i \in \Phi} \in E^{F \Phi}_T(V, X) | x_i = 0 \text{ for } 0 \neq i \in \Phi, x_0 \text{ factors through add}(M) \text{ and } w[-1]\}$.

(3) For $x = (x_i)_{i \in \Phi} \in E^{F \Phi}_T(V', X)$ with $V' \in \text{add}(V)$, we have Im($\mu(x)$) $\subseteq I \cdot E^{F \Phi}_T(V, X)$ if and only if $x_i = 0$ for all $0 \neq i \in \Phi$ and $x_0$ factors through add($M$) and $w[-1]$, where $\mu$ is defined in Lemma 2.2 (1).

(4) Let $f : M' \to X$ with $M' \in \text{add}(M)$. Then Im($E^{F \Phi}_T(V, f)$) $\subseteq I \cdot E^{F \Phi}_T(V, X)$ if and only if $f$ factors through $w[-1]$.

Proof. (1) We denote by $\lambda_M = (0, 1) : M \to V$ the canonical inclusion. Let $(x_i)_{i \in \Phi} \in I$ and $(y_i)_{i \in \Phi} \in E^{F \Phi}_T(V, M)$. Then $(x_i)(y_i) = (x_0y_i)_{i \in \Phi}$ since $x_i = 0$ for $0 \neq i \in \Phi$. Since $I$ is an ideal in $E^{F \Phi}_T(V)$, it follows that $x_0(y_i (F^x \lambda_M))_{i \in \Phi} = (x_0y_i (F^x \lambda_M))_{i \in \Phi} \in I$. By the definition of $I$, we have $x_0y_i (F^x \lambda_M) = 0$ for all $0 \neq i \in \Phi$ and $x_0y_0 \lambda_M$ factorises through $w[-1]$. Moreover, $x_0y_0 \lambda_M = (x_0y_0 \lambda_M) \lambda_M = 0$.
\( s(w[-1]|pM) \lambda_M = s \cdot \lambda_M = 0 \), where \( s \) is a morphism from \( V \) to \( Y[-1] \). Hence \( x_0 y_i (F^i \lambda_M) = 0 \) for all \( i \in \Phi \). Thus (1) follows.

(2) Let \( \lambda_X : X \to V \) be the canonical inclusion. As in case (1), it follows that, for \((x_i)_{i \in \Phi} \in I\) and \((y_i)_{i \in \Phi} \in \mathbb{E}^F_{/\Phi}(V, X)\), we have \((x_i)(y_i) = (x_0 y_i)_{i \in \Phi}\), and \( x_0 y_0 \lambda_X \) factorizes through \( \tilde{w}[1] \) and \( \text{add}(M) \). Hence \( x_0 y_0 = (x_0 y_0 \lambda X) [pX] = s(\pi [1]) pX = s \), where \( s \) is a morphism from \( V \) to \( Y[-1] \). Conversely, let \( x = \lambda X [pX] \in \mathbb{E}^F_{/\Phi}(V, X) \) and suppose that \( x_0 = 0 \) for all \( 0 \neq i \in \Phi \) and that \( x_0 \) factorizes through \( \text{add}(M) \) and \( \pi[1] \). For \( f : U \to Z \) in \( T \), we denote by \( f \) the element of \( \mathbb{E}^F_{/\Phi}(U, Z) \) concentrated only in degree \( 0 \in \Phi \). Then it is straightforward to check that \( \lambda X [x] \) belongs to \( I \). Thus, \( \lambda_X (x) \) belongs to \( I \cdot \mathbb{E}^F_{/\Phi}(V, X) \). Hence \((x_0 y_i)_{i \in \Phi} \) does not factorize through \( \text{add}(M) \) and \( \pi[1] \). Conversely, suppose that \( x \in I \cdot \mathbb{E}^F_{/\Phi}(V, X) \). Then, for any \( y \in \mathbb{E}^F_{/\Phi}(V) \), the image of \( y \) under \( \mu(x) \) is \( y \cdot x \). Since \( I \cdot \mathbb{E}^F_{/\Phi}(V, X) \) is a \( \mathbb{E}^F_{/\Phi}(V, X) \)-submodule of \( \mathbb{E}^F_{/\Phi}(V, X) \), we have \( y \cdot x \in I \cdot \mathbb{E}^F_{/\Phi}(V, X) \).

Secondly, suppose that \( V' \) is a direct sum of \( n \) copies of \( V \), \( x \in \mathbb{E}^F_{/\Phi}(V', X) \). We identify \( \mathbb{E}^F_{/\Phi}(V', X) \) with \( \bigoplus_{i=1}^n \mathbb{E}^F_{/\Phi}(V, X) \), and write \( x = (a_1, \cdots, a_n) \), a column matrix with \( a_i \in \mathbb{E}^F_{/\Phi}(V, X) \). Then the image of \( \mu(x) \) is the sum of the image of \( \mu(a_i) \) for \( 1 \leq i \leq n \). Now the conclusion follows from the first case.

Finally, suppose that \( V' \) is a direct summand of \( n \) copies of \( V \), that is, \( \bigoplus_{i=1}^n V = V' \oplus V'' \). If \( x \in \mathbb{E}^F_{/\Phi}(V', X) \), then we may consider \( (x, 0) \) as an element in \( \mathbb{E}^F_{/\Phi}(\bigoplus_{i=1}^n V, X) \). Then the proof is reduced to the second case.

(4) follows from (3) because of \( \mathbb{E}^F_{/\Phi}(V, f) = \mu(f) \). \( \square \)

Let \( \tilde{T}^* \) be the complex

\[
\tilde{T}^*: \quad 0 \longrightarrow \mathbb{E}^F_{/\Phi}(V, X) \xrightarrow{E^F_{/\Phi}(V, \alpha)} \mathbb{E}^F_{/\Phi}(V, M_1 \oplus M) \longrightarrow 0,
\]

where the term \( \mathbb{E}^F_{/\Phi}(V, X) \) is in degree zero. Then it is the direct sum of the following two complexes

\[
0 \longrightarrow \mathbb{E}^F_{/\Phi}(V, X) \xrightarrow{E^F_{/\Phi}(V, \alpha)} \mathbb{E}^F_{/\Phi}(V, M_1) \longrightarrow 0,
\]

\[
0 \longrightarrow 0 \longrightarrow \mathbb{E}^F_{/\Phi}(V, M) \longrightarrow 0.
\]

Let \( P = \mathbb{E}^F_{/\Phi}(V, X)/I \cdot \mathbb{E}^F_{/\Phi}(V, X) \), and let \( p : \mathbb{E}^F_{/\Phi}(V, X) \to P \) be the canonical surjection. Then, by Lemma 3.4 (1), we may write \( \mathbb{E}^F_{/\Phi}(V, \alpha) = \beta P \) with \( \beta : P \to \mathbb{E}^F_{/\Phi}(V, X) \). The complex

\[
T^*: \quad 0 \longrightarrow P \longrightarrow \mathbb{E}^F_{/\Phi}(V, M_1 \oplus M) \longrightarrow 0
\]

in \( \mathbb{E}^F_{/\Phi}(V)/I \) is the direct sum of the complexes

\[
0 \longrightarrow P \xrightarrow{q} \mathbb{E}^F_{/\Phi}(V, M_1) \longrightarrow 0,
\]

\[
0 \longrightarrow 0 \longrightarrow \mathbb{E}^F_{/\Phi}(V, M) \longrightarrow 0.
\]

Each term of \( T^* \) is a finitely generated projective \( \mathbb{E}^F_{/\Phi}(V)/I \)-module.

Before proceeding further, we need to introduce some more notation. Set

\[
\Lambda := \mathbb{E}^F_{/\Phi}(V), \quad \Gamma := \mathbb{E}^F_{/\Phi}(W), \quad \tilde{\Lambda} := \Lambda / I, \quad \tilde{\Gamma} := \Gamma / J,
\]

where \( I \) and \( J \) are defined just before Theorem 3.1.
Lemma 3.5. $T^*$ is a tilting complex over $\Lambda$.

Proof. It is clear that $\text{Hom}_{\text{add}(\Lambda)}(T^*, T^*[i]) = 0$ for $i \leq -2$ and for $i \geq 2$. We have to check that $\text{Hom}_{\text{add}(\Lambda)}(T^*, T^*[1]) = 0$ and $\text{Hom}_{\text{add}(\Lambda)}(T^*, T^*[-1]) = 0$.

In the following, for a morphism $f^*$ between complexes $U^*$ and $V^*$, we write $[f^*]$ for the class of $f^*$ in the homotopy category.

Let $[f^*] \in \text{Hom}_{\text{add}(\Lambda)}(T^*, T^*[1])$. Consider the following diagram:

$$
\begin{array}{ccc}
E^{F\Phi}_T(V, X) & \xrightarrow{E^{F\Phi}_T(V, \alpha)} & E^{F\Phi}_T(V, M_1 \oplus M) \\
\downarrow{\mu(u)} & & \downarrow{\mu(u')} \\
E^{F\Phi}_T(V, M_1 \oplus M) & \xrightarrow{E^{F\Phi}_T(V, \alpha)} & E^{F\Phi}_T(V, M_1 \oplus M)
\end{array}
$$

In fact, if $a = (a_j)_{j \in \Phi} \in E^{F\Phi}_T(V, X)$, then it is sent to $b := (a_j F^j(\alpha))_{j \in \Phi} \in E^{F\Phi}_T(V, \alpha)$, and further sent to $bu' = (a_j F^j(\alpha))_{j \in \Phi} u'$ by $\mu(u')$. An easy calculation shows that $bu' = au$, the image of $a$ under $\mu(u)$. Thus the diagram is commutative, and

$$
pf^0 = \mu(u) = E^{F\Phi}_T(V, \alpha)\mu(u') = pq\mu(u').
$$

This means that $f^0 = q\mu(u')$ - since $p$ is surjective - and that $[f^*] = 0$ in $\text{add}(\Lambda)$. Therefore $\text{Hom}_{\text{add}(\Lambda)}(T^*, T^*[1]) = 0$.

Let $[f^*] \in \text{Hom}_{\text{add}(\Lambda)}(T^*, T^*[-1])$. Consider the following diagram:

$$
\begin{array}{ccc}
0 & \xrightarrow{q} & E^{F\Phi}_T(V, M_1 \oplus M) \\
\downarrow & & \downarrow{f^1} \\
0 & \xrightarrow{q} & E^{F\Phi}_T(V, M_1 \oplus M)
\end{array}
$$

Since $p$ is surjective and $E^{F\Phi}_T(V, M_1 \oplus M)$ is projective in $\Lambda$-mod, $f^1$ can be lifted along $p$, say $f^1 = gp$ with $g : E^{F\Phi}_T(V, M_1 \oplus M) \to E^{F\Phi}_T(V, X)$. By assumption, we have $X \in \text{add}(M)$, and, by Lemma 2.2 (3), there is a homomorphism $u : M_1 \oplus M \to X$ such that $g = E^{F\Phi}_T(V, u)$. Thus

$$
E^{F\Phi}_T(V, u\alpha) = E^{F\Phi}_T(V, u)E^{F\Phi}_T(V, \alpha) = gpq = f^1q = 0.
$$

Lemma 2.2 (2) implies $u\alpha = 0 = u\alpha$. Therefore $u$ factorises through $-w[-1]$. By Lemma 3.4 (4), the image of $g = E^{F\Phi}_T(V, u)$ is contained in $I \cdot E^{F\Phi}_T(V, X)$. It follows that $f^1 = gp = 0$ and $[f^*] = 0$. Hence $\text{Hom}_{\text{add}(\Lambda)}(T^*, T^*[-1]) = 0$. 

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Now, $\text{add}(T^*)$ generates $\mathcal{X}^b(\Lambda\text{-proj})$ as a triangulated category. Thus $T^*$ is a tilting complex over $\Lambda$. □

Remark. To get a tilting complex from $\tilde{T}^*$, one may consider the ideal $I_0$ of $E^F_\Phi(V)$ consisting of all endomorphisms $V \to V$ which are of the form $f \circ g$ with $f : V \to M'$ and $g : M' \to V$ such that $M' \in \text{add}(M)$ and $g\tilde{\alpha} = 0$. Then it is easy to show that the quotient complex of $\tilde{T}^*$ modulo $I_0 \tilde{T}^*$ is a two-term tilting complex over $E^F_\Phi(V)/I_0$. We will not use this complex because its endomorphism algebra cannot be described in a nice way. Note that the ideal $I_0$ of $E^F_\Phi(V)$ is properly contained in $I$ in general.

**Lemma 3.6.** The two rings $\overline{\Lambda}$ and $\text{End}_{\mathcal{X}^b(\overline{\Lambda}\text{-proj})}(T^*)$ are isomorphic.

**Proof.** Since $\overline{\Lambda}$ is a quotient algebra of $\Lambda$, the category $\overline{\Lambda}\text{-mod}$ can be viewed as a full subcategory of $\Lambda\text{-mod}$, and it follows that $\mathcal{X}^b(\overline{\Lambda})$ can be viewed as a full subcategory of $\mathcal{X}^b(\Lambda)$. Thus, we have an isomorphism $\text{End}_{\mathcal{X}^b(\overline{\Lambda}\text{-proj})}(T^*) \simeq \text{End}_{\mathcal{X}^b(\Lambda)(T^*)}$. To prove the lemma, we shall construct an isomorphism from $\text{End}_{\mathcal{X}^b(\Lambda)(T^*)}$ to $\overline{\Lambda}$.

Let $[f^*] \in \text{End}_{\mathcal{X}^b(\Lambda)(T^*)}$. Since $p : E^F_\Phi(V, X) \to P$ is an epimorphism and $E^F_\Phi(V, X)$ is a projective $\Lambda$-module, there is a $\Lambda$-module homomorphism $u^0 : E^F_\Phi(V, X) \to E^F_\Phi(V, X)$ such that $u^0 p = p f^0$. Let $u^1 := f^1$ and $u^0 = 0$ for all $i \neq 0$. Then it follows from

$$u^0 E^F_\Phi(V, \tilde{\alpha}) = u^0 pq = p f^0 q = p q f^1 = E^F_\Phi(V, \tilde{\alpha}) u^1$$

that $u^* = (u^i)_{i \in \mathbb{Z}}$ is an endomorphism in $\text{End}_{\mathcal{X}^b(\Lambda)(T^*)}$. By Lemma 2.2 (1), we can assume that $u^0 = \mu(x)$ and $u^1 = \mu(y)$ with $x = (x_i)_{i \in \Phi} \in E^F_\Phi(X)$ and $y = (y_i)_{i \in \Phi} \in E^F_\Phi(M_1 + M)$. Now, it follows from $E^F_\Phi(V, \tilde{\alpha}) u^1 = u^0 E^F_\Phi(V, \tilde{\alpha})$ that

$$(\tilde{\alpha} x_i)_{i \in \Phi} = (x_i F^i \tilde{\alpha})_{i \in \Phi},$$

that is, $\tilde{\alpha} x_i = x_i F^i \tilde{\alpha}$ for $i \in \Phi$.

For each $i \in \Phi$, we can form the following commutative diagram in $\mathcal{T}$:

$$
\begin{array}{ccccccccc}
X & \xrightarrow{\alpha} & M_1 + M & \xrightarrow{\beta} & W & \xrightarrow{\bar{\psi}} & X[1] \\
\downarrow x_i & & \downarrow y_i & & \downarrow h_i & & \downarrow x_i[1] \\
F^iX & \xrightarrow{F^i\alpha} & F^i(M_1 + M) & \xrightarrow{F^i\beta} & F^iW & \xrightarrow{(F^i\bar{\psi})(F^i,x,\mathbf{1})} & (F^iX)[1],
\end{array}
$$

for some morphism $h_i \in \text{Hom}_T(W, F^i W)$. Thus, for each $[f^*] \in \text{End}_{\mathcal{X}^b(\Lambda)(T^*)}$, we get an element $h := (h_i)_{i \in \Phi} \in \Gamma$ which is $E^F_\Phi(W)$ by definition. This leads us to defining the following correspondence:

$$\Theta : \text{End}_{\mathcal{X}^b(\Lambda)(T^*)} \to \overline{\Lambda} = \Gamma/J, \quad [f^*] \mapsto h + J.$$

**Claim 1.** $\Theta$ is well defined.

**Proof.** Suppose that $[f^*] \in \text{End}_{\mathcal{X}^b(\Lambda)(T^*)}$ is null-homotopic, that is, there is a map $r : E^F_\Phi(V, M_1 + M) \to P$ such that $f^0 = qr$ and $f^1 = rq$. Since $p$ is surjective and $E^F_\Phi(V, M_1 + M)$ is projective in $\Lambda\text{-mod}$, there is a map $s : E^F_\Phi(V, M_1 + M) \to E^F_\Phi(V, X)$ such that $s p = r$. Hence $(u^0 - pqs) p = u^0 p - pq s = u^0 p - pq r = u^0 p - p f^0 = 0$ and $u^1 = qr = sp q$. By the assumption $X \in \mathcal{X}^b(\Phi)$, Lemma 2.2 (3) yields a map $t : M_1 + M \to X$ such that $s = E^F_\Phi(V, t) = \mu(t)$. Therefore,

$$
\mu(x - \tilde{\alpha} t) = (u^0 - E^F_\Phi(V, \tilde{\alpha}) E^F_\Phi(V, t)) p = (u^0 - pqs) p = 0
$$

and $\mu(y - t \tilde{\alpha}) = u^1 - sp q = 0$. Consequently, $\text{Im}(\mu(x - \tilde{\alpha} t)) \subseteq I \cdot E^F_\Phi(V, X)$ and $y - t \tilde{\alpha} = 0$. Thus $y_i = 0$ for all $0 \neq i \in \Phi$ and $y_0 = t \tilde{\alpha}$. By Lemma 3.4 (3), we have $x_i = 0$ for all $0 \neq i \in \Phi$ and $x_0 - \tilde{\alpha} t = ab$.
for some morphisms $a : X \to M'$ and $b : M' \to X$ with $M' \in \text{add}(M)$. Since $\alpha$ is a left $\text{add}(M)$-approximation of $X$, there is a morphism $c : M_1 \oplus M \to M'$ such that $a = \alpha c$. It follows that

$$x_0 = ab + \alpha a = \alpha cb + \alpha t = \alpha(cb + t).$$

Now we consider the commutative diagram (\ast). Suppose $0 \neq i \in \Phi$. Then we have shown that $x_i = y_i = 0$. Hence $\beta h_i = y_i F^i \beta = 0$. This implies that $h_i$ factorizes through $\tilde{\nu}$, and, consequently, that $h_i \mid M = 0$ since $\tilde{\nu} \mid M = 0$. It follows from $h_i(F^i \delta(F,i,X,1)) = \tilde{\nu}(x_i[1]) = 0$ that $h_i : W \to F^i W$ factorizes through $F^i(M_1 \oplus M)$. Since $Y \in \mathcal{E}_{F^i \Phi}(M)$, we get $h_i \mid Y = 0$. Altogether, we have shown that $h_i = 0$ for all $0 \neq i \in \Phi$. Now consider the diagram (\ast) in case $i = 0$. First, we have $\beta h_0 = y_0 \beta = \tau \alpha \beta = 0$, which means $h_0$ factorizes through $\tilde{\nu}$. Second, since $h_0 \tilde{\nu} = \tilde{\nu}(x_0[1]) = \tilde{\nu}(\alpha[1])(cb + t)[1] = 0$, the morphism $h_0$ factorizes through $M_1 \oplus M$ which is in $\text{add}(M)$. Thus, $h \in J$ and $h + J$ is zero in $\mathbb{T}$. This shows that $\Theta$ is well-defined.

Claim 2. $\Theta$ is injective.

Proof. Suppose that $\Theta([\ast]) = h + J = 0 + J$. Then $h \in J$, that is, $h_i = 0$ for all $0 \neq i \in \Phi$, and $h_0$ factorizes through both $\tilde{\nu}$ and $\text{add}(M)$. Suppose $h_0 = \tilde{\nu} s$ for a morphism $s : X[1] \to W$. For each $0 \neq i \in \Phi$, since $y_i F^i \beta = \beta h_i = 0$, the morphism $y_i : M_1 \oplus M \to F^i(M_1 \oplus M)$ factorizes through $F^i X$, and consequently $y_i = 0$ for all $0 \neq i \in \Phi$ since $X \in \mathcal{E}_{F^i \Phi}(M)$. For each $0 \neq i \in \Phi$, it follows from $\tilde{\nu}(x_i[1]) = h_i(F^i \delta(F,i,X,1)) = \tilde{\nu}(x_i[1]) = 0$ that $x_i[1]$ factorizes through $(M_1 \oplus M)[1]$, or equivalently, the morphism $x_i : X \to F^i X$ factorizes through $\tilde{\nu}$.

Hence $x_i = 0$ for all $0 \neq i \in \Phi$. Now we consider the case $i = 0$. First, we have $y_0 \tilde{\nu} = \beta h_0 = \beta \tilde{\nu} \beta = 0$, which implies $y_0 = \tau \alpha \beta$ for a morphism $t : M_1 \oplus M \to X$. Second, $(x_0 - \alpha t) \alpha = (x_0 - \alpha t) \alpha = 0$. It follows that $(x_0 - \alpha t) \alpha = 0$, and therefore $x_0 - \alpha t$ factorizes through $-w[-1]$. Since $h_0 : W \to W$ factorizes through $\text{add}(M)$ and since $\beta : M_1 \oplus M \to W$ is a right $\text{add}(M)$-approximation of $W$, we see that $h_0$ factorizes through $\tilde{\nu}$, say $h_0 = r \tilde{\nu}$ for some $r : W \to M_1 \oplus M$. Thus, $\tilde{\nu}(x_0[1]) = h_0 \tilde{\nu} = r \tilde{\nu} = 0$, or equivalently, $(-w[-1]) x_0 = 0$. It follows that $x_0$ factorizes through $M_1 \oplus M$. Since $\alpha \beta$ also factorizes through $M_1 \oplus M$, we see that $x_0 - \alpha t$ factorizes through $\text{add}(M)$. Thus we have shown that $x_0 - \alpha t$ factorizes through both $\text{add}(M)$ and $-w[-1]$. Now, by Lemma 3.4 (3), we have $\text{Im}(\mu(x) - E_{F^i \Phi}(V, \alpha t)) = \text{Im}(\mu(x) - \tilde{\nu} \beta t) \subseteq I \cdot E_{F^i \Phi}(V,X)$. Hence

$$p(f^0 - q E_{F^i \Phi}(V,t)p) = u^0 p - pq E_{F^i \Phi}(V,t)p = (\mu(x) - E_{F^i \Phi}(V, \alpha t)) p = 0.$$

This implies that $f^0 = q(E_{F^i \Phi}(V,t)p)$ since $p$ is surjective. Moreover, one can check that

$$f^1 = u^1 = \mu(y) = E_{F^i \Phi}(V,t) E_{F^i \Phi}(V,\alpha) = (E_{F^i \Phi}(V,t)p)q.$$

Hence $\ast$ is null-homotopic, and consequently $\Theta$ is injective.

Claim 3. $\Theta$ is surjective.

Proof. Let $h = (h_i)_{i \in \Phi} \in \Gamma$ with $h_i : W \to F^i W$ for $i \in \Phi$. Since $\tilde{\nu}$ is a right $(\text{add}(M), F, -\Phi)$-approximation of $W$, we have a morphism $F^{-i}y_i : F^{-i}(M_1 \oplus M) \to M_1 \oplus M$ such that $(F^{-i} \tilde{\nu})(F^{-i} h_i) = (F^{-i} y_i) \beta$ for $i \in \Phi$. This means that there is a commutative diagram

$\begin{array}{cccc}
X & \xrightarrow{\tilde{\nu}} & M_1 \oplus M & \xrightarrow{\beta} & W & \xrightarrow{\tilde{\nu}} & X[1] \\
\downarrow x_i & & \downarrow y_i & & \downarrow h_i & & \downarrow x_i[1] \\
F^i X & \xrightarrow{F^i \tilde{\nu}} & F^i(M_1 \oplus M) & \xrightarrow{F^i \beta} & F^i W & \xrightarrow{(F^i \alpha)(F,i,X,1)} & F^i X[1]
\end{array}$

Now, define $x := (x_i)_{i \in \Phi} \in E_{F^i \Phi}(X)$, $y := (y_i)_{i \in \Phi} \in E_{F^i \Phi}(M_1 \oplus M)$; $u^0 := \mu(x)$, $u^1 := \mu(y)$ and $u^j := 0$ for $j \neq 0, 1$. Then $u^* := (u^j)_{j \in \mathbb{Z}}$ belongs to $\text{End}_{\mathcal{E}_{F^i \Phi}(X)}(\mathcal{T})$. Since $u^* : E_{F^i \Phi}(V,X) \to E_{F^i \Phi}(V,X)$ takes elements in $I \cdot E_{F^i \Phi}(V,X)$ to elements in $I \cdot E_{F^i \Phi}(V,X)$, the image of $I \cdot E_{F^i \Phi}(V,X)$ under the map $u^0 p$ is
Proof. The map \( \Theta \) is \( R \)-linear, so it preserves addition. For multiplication, we take \([f^*] \) and \([g^*] \) in \( \text{End}_X^{\psi_{(\Lambda)}}(T^*) \). Let \([u^*] \) and \([v^*] \) be in \( \text{End}_X^{\psi_{(\Lambda)}}(T^*) \) such that \( u^0p = pf^0, \ u^1 = f^1, \ v^0p = pg^0 \) and \( v^1 = g^1 \). Suppose that \((u^0, u^1) = (\mu(x), \mu(y)) \) and \((v^0, v^1) = (\mu(x'), \mu(y')) \) with \( x, x' \in E_T^0(X) \) and \( y, y' \in E_T^0(M_1 \oplus M) \). Let \( h := (h_i)_{i \in \Phi} \) and \( h' := (h'_i)_{i \in \Phi} \) be in \( \Gamma \) making the diagram \((*) \) commutative, that is,
\[
\bar{\beta} * h_i = \bar{\beta} h_i = y_i F^i \bar{\beta} = y_i' \bar{\beta}, \quad \bar{w}(x_i[1]) = h_i(F^i \bar{w}) \delta(F, i, X, 1) = (h_i * \bar{w}) \delta(F, i, X, 1),
\]
and
\[
\bar{\beta} * h'_i = \bar{\beta} h'_i = y'_i F^i \bar{\beta} = y'_i \bar{\beta}, \quad \bar{w}(x'_i[1]) = h'_i(F^i \bar{w}) \delta(F, i, X, 1) = (h'_i * \bar{w}) \delta(F, i, X, 1),
\]
for all \( i \in \Phi \). Then, by definition, we have \( \Theta([f^*]) = h + J, \Theta([g^*]) = h' + J \) and
\[
\Theta([f^*]) \Theta([g^*]) = \left( \sum_{i \in \Phi} h_i * h'_i \right)_{k \in \Phi} + J.
\]
Now we calculate \( \Theta([f^*g^*]) \). Let \( s^* := u^*v^* \). Then \( s^0p = pf^0g^0 = p(f^*g^*)^0, \ s^1 = f^1g^1 = (f^*g^*)^1 \), and \((s^0, s^1) = (\mu(xx'), \mu(yy'))\), where \((xx')_k = \sum_{i \in \Phi} x_i * x'_j \) and \((yy')_k = \sum_{i \in \Phi} y_i * y'_j \). For each \( k \in \Phi \), we have
\[
(yy')_k F^k \bar{\beta} = (yy')_k * \bar{\beta} = \left( \sum_{i \in \Phi} y_i * y'_j \right) * \bar{\beta} = \bar{\beta} * \left( \sum_{i \in \Phi} h_i * h'_j \right).
\]
Similarly, for each \( k \in \Phi \), we have
\[
\left( \sum_{i \in \Phi} h_i * h'_j \right)(F^k \bar{w}) \delta(F, k, X, 1) = \bar{w}(\left(\sum_{i \in \Phi} y_i * y'_j\right)[1]).
\]
This means that \( \Theta([f^*g^*]) = \Theta([f^*g^*]) = \left( \sum_{i \in \Phi} h_i * h'_j \right)_{k \in \Phi} + J = \Theta([f^*]) \Theta([g^*]) \). Thus \( \Theta \) is a ring homomorphism, and the proof of Theorem 3.1 is finished. \( \square \)

Before we proceed, we comment on the conditions in Theorem 3.1.

(a) Let \( X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{\gamma} X[1] \) be a triangle in \( T \) with \( M_1 \in \text{add}(M), \ X \in \mathscr{A}^F \Phi(M) \) and \( Y \in \mathscr{A}^F \Phi(M) \). If \( \alpha \) is a left \( \text{add}(M), F, \Phi \)-approximation of \( X \), then \( \text{Hom}_T(X, F^i M) \simeq \text{Hom}_T(M_1, F^i M) \) for \( 0 \neq i \in \Phi \). Similarly, if \( \beta \) is a right \( \text{add}(M), F, -\Phi \)-approximation of \( Y \), then \( \text{Hom}_T(M, F^i Y) \simeq \text{Hom}_T(M, F^i M_1) \) for \( 0 \neq i \in \Phi \). In particular, if \( M \) is an \( (F, \Phi) \)-self-orthogonal object of \( T \), that is, \( \text{Hom}_T(M, F^i M) = 0 \) for every \( 0 \neq i \in \Phi \), and if \( \alpha \) is a left \( \text{add}(M), F, \Phi \)-approximation of \( X \) and \( \beta \) is a right \( \text{add}(M), F, -\Phi \)-approximation of \( Y \), then \( X \in \mathscr{A}^F \Phi(M) \) and \( Y \in \mathscr{A}^F \Phi(M) \).

(b) Under the conditions of Theorem 3.1, there are isomorphisms \( \text{Hom}_T(X, F^i X) \simeq \text{Hom}_T(Y, F^i Y) \) for every \( 0 \neq i \in \Phi \). In fact, this follows from the following general statement:

Let \( T \) be a triangulated category with a shift functor \([1] \). Suppose that \( F \) is a triangle functor from \( T \) to itself, and that \( D \) is a full subcategory of \( T \). Let \( i \) be a positive integer. Suppose that
\[
X_j \xrightarrow{\alpha_j} D_j \xrightarrow{\beta_j} Y_j \xrightarrow{\gamma_j} X_j[1]
\]
is a triangle in $\mathcal{T}$, such that $\alpha_j$ is a left $(\mathcal{D}, F, \{i\})$-approximation of $X_j$, and that $\text{Hom}_T(D', F^i(\beta_j)) : \text{Hom}_T(D', F^i(Y)) \to \text{Hom}_T(D', F^i(Y_j))$ is surjective for every $D' \in \mathcal{D}$ and $j = 1, 2$. If $\text{Hom}_T(D, F^i(X)) = 0 = \text{Hom}_T(Y_j, F^i(D))$ for $1 \leq j \leq 2$, then $\text{Hom}_T(X_1, F^iX_2) \simeq \text{Hom}_T(Y_1, F^iY_2)$.

Proof. From the given two triangles the following exact commutative diagram can be formed:

\[
\begin{array}{cccc}
\text{Hom}_T(D_1, F^iX_2) & \longrightarrow & \text{Hom}_T(D_1, F^iD_2) \\
\downarrow & & \downarrow (\alpha_1, F^iD_2) \\
\text{Hom}_T(X_1, F^iX_2) & \longrightarrow & \text{Hom}_T(X_1, F^iD_2) & 0 \\
\downarrow & & \downarrow \\
\text{Hom}_T(Y_1, F^iD_2) & \longrightarrow & \text{Hom}_T(Y_1, F^iY_2) & \longrightarrow \text{Hom}_T(Y_1, F^iX_2[1]) & \longrightarrow \text{Hom}_T(Y_1, F^iD_2[1]) \\
\downarrow & & \downarrow (\ast) & & \downarrow \\
\text{Hom}_T(D_1, F^iD_2) & \longrightarrow & \text{Hom}_T(D_1, F^iY_2) & \longrightarrow \text{Hom}_T(D_1, F^iX_2[1]) & \longrightarrow \text{Hom}_T(D_1, F^iD_2[1])
\end{array}
\]

Since $\text{Hom}_T(Y_1, F^iD_2) = \text{Hom}_T(D_1, F^iX_2) = 0$ by assumption and since $\text{Hom}_T(\alpha_1, F^iD_2)$ and $\text{Hom}_T(D_1, F^i\beta_2)$ are surjective by the property of approximation, the conclusion follows from the commutative square $(\ast)$. □

(c) Let $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ be an (add($M$))-split triangle in $\mathcal{T}$. Define $V := X \oplus M$, $\Lambda_0 := \text{End}_T(V)$, $W := M \oplus Y$, and $\Gamma_0 := \text{End}_T(W)$. Let $I$ and $J$ be as defined in Theorem 3.1. Then the ideals $I$ and $J$ in Theorem 1.1 have the following characterisation:

(i) Let $e$ be the idempotent in $\Gamma_0$ corresponding to the direct summand $M$ of $W$. Then $J$ is the submodule of the left $\Gamma_0$-module $\Gamma_0 \oplus \Gamma_0$, which is maximal with respect to $eJ = 0$.

(ii) Let $f$ be the idempotent in $\Lambda_0$ corresponding to the direct summand $M$ of $V$. Then $I$ is the submodule of the right $\Lambda_0$-module $\Lambda_0 \oplus \Lambda_0$ which is maximal with respect to $If = 0$.

Proof. By Lemma 3.3, the sets $I$ and $J$ are ideals of $\Lambda_0$ and $\Gamma_0$, respectively.

(i) Let $p_M : W \to M$ and $\lambda_M : M \to W$ be the canonical projection and injection, respectively. By definition, $e = p_M \lambda_M$. The set $\Gamma_0 \oplus \Gamma_0$ is precisely the set of all endomorphisms of $W$ that factorise through add($M$). The endomorphisms of $W$ factoring through $w$ are those endomorphisms $x$ that satisfy $\beta x = 0$, and consequently $ex = p_M \lambda_M x = p_M (\beta|_M x) = 0$. Hence $J$ is a submodule of $\Gamma_0 \Gamma_0 \oplus \Gamma_0$ with $eJ = 0$. Suppose that $\tilde{J} \subseteq \Gamma_0 \Gamma_0 \oplus \Gamma_0$ is another submodule containing $J$ with $e\tilde{J} = 0$. Then $e\tilde{J} = 0$ implies $\text{Hom}_T(\tilde{J}, W) = 0$, and consequently $\text{Hom}_T(J, W, M') = 0$ for all $M' \in \text{add}(M)$. For each $x \in J$, the image of the morphism $\text{Hom}_T(W, x)$ is contained in $\tilde{J}$ since $\tilde{J}$ is a left ideal of $\Gamma_0$. Thus, the morphism $\text{Hom}_T(W, \beta x)$ is a $\Gamma_0$-module morphism from $\text{Hom}_T(W, M_1 \oplus M)$ to the image of $\text{Hom}_T(W, x)$. Hence $\text{Hom}_T(W, \beta x) = 0$, and consequently $\beta x = 0$. This implies $x \in J$. This proves (i).

(ii) The proof is similar to that of (i). □

Note that if $\text{Hom}_T(X, F^iD') = 0$ for all $0 \neq i \in \Phi$ and $D' \in \mathcal{D}$, then $f$ is a left $(\mathcal{D}, F, \Phi)$-approximation of $X$ if and only if $f$ is a left $\mathcal{D}$-approximation of $X$. A dual statement is also true for a right $(\mathcal{D}, F, \Phi)$-approximation of $X$. Thus, a special case of Theorem 3.1 is the following corollary for $\mathcal{D}$-split triangles (see Section 2.4 for definition).

**Corollary 3.7.** Let $\Phi$ be an admissible subset of $\mathbb{Z}$, and let $\mathcal{T}$ be a triangulated $R$-category with a triangle auto-equivalence $F$, and let $M$ be an object in $\mathcal{T}$. Suppose that $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ is an add($M$)-split triangle in $\mathcal{T}$, and suppose that $X$ and $Y$ both are in $\mathcal{X}^{-F, \Phi}(M) \cap \mathcal{Y}^{-F, \Phi}(M)$. Then $E_{\mathcal{T}}^{F, \Phi}(X \oplus M)/I$ and $E_{\mathcal{T}}^{F, \Phi}(M \oplus Y)/J$ are derived equivalent.

The following special case of Theorem 3.1 is useful to construct explicit examples.
Corollary 3.8. Let $T$ be a triangulated $R$-category with $[1]$ the shift functor, and let $M$ be an object in $T$. Suppose that $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ is a triangle in $T$ such that $M_1 \in \text{add}(M)$, and suppose that $X \in \mathcal{D}_{n+1}(M)$ and $Y \in \mathcal{D}_{n+1}(M)$. Then, for any admissible subset $\Phi$ of $\mathbb{N}_n$, the algebras $E^\Phi_T(X \oplus M)/I$ and $E^\Phi_T(M \oplus Y)/J$ are derived equivalent.

Proof. We show that $\beta$ is a right $(\text{add}(M), -\Phi)$-approximation of $Y$. Note that, for $i \in \Phi$, we always have $i + 1 \leq n + 1$. Hence $\text{Hom}_T(M, X[i+1]) = 0$ for $i \in \Phi$. Now apply $\text{Hom}_T(M[-i], -)$ with $i \in \Phi$ to the triangle $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$:

$$\cdots \to \text{Hom}_T(M[-i], M_1) \to \text{Hom}_T(M[-i], Y) \to \text{Hom}_T(M[-i], X[1]) \to \cdots$$

Because of $\text{Hom}_T(M[-i], X[1]) = \text{Hom}_T(M, X[i+1]) = 0$, the map $\beta$ is a right $(\text{add}(M), -\Phi)$-approximation of $Y$.

Similarly, it follows from $\text{Ext}_T^{i+1}(Y, M) = 0$ for $i \in \Phi$ that $\alpha$ is a left $(\text{add}(M), \Phi)$-approximation of $X$. Now Corollary 3.8 follows from Theorem 3.1. □

Another special case of Theorem 3.1 is that $I = 0$ and $J = 0$. Here is a condition when the ideals $I$ and $J$ in Theorem 3.1 vanish.

Proposition 3.9. Let $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ be an $(\text{add}(M))$-split triangle in $T$. Define $V := X \oplus M$, $\Lambda_0 := \text{End}_T(V)$, $W := M \oplus Y$, and $\Gamma_0 := \text{End}_T(W)$. Let $I'$ be the ideal of $\Lambda_0$ consisting of all $f : V \to V$ that factorises through $\tilde{\omega}[-1] : Y[-1] \to V$, and let $J'$ be the ideal of $\Gamma_0$ consisting of all $g : W \to W$ that factorises through $\tilde{w} : W \to X[1]$.

1. Suppose that $\Lambda_0$ is an Artin algebra. If $\text{add}(\text{top}_{\Lambda_0} \text{Hom}_T(V, X)) \cap \text{add}(\text{top}_{\Lambda_0} \text{D}\Lambda_0)) = 0$, then $I' = 0$.
2. Suppose that $\Gamma_0$ is an Artin algebra. If $\text{add}(\text{top}_{\Gamma_0} \text{Hom}_T(W, Y)) \cap \text{add}(\text{soc}(\Lambda_0) \Gamma_0)) = 0$, then $J' = 0$.

By definition, there are inclusions $I \subseteq I'$ and $J \subseteq J'$. Sometimes it is easy to verify that $I'$ and $J'$ vanish if the algebras $\Lambda_0$ and $\Gamma_0$ are described by quivers with relations.

Proof of Proposition 3.9. We prove (1). The proof of (2) is similar to that of (1), and we omit it.

We have a triangle $Y[-1] \xrightarrow{-\tilde{\omega}[-1]} V \xrightarrow{-\alpha} M_1 \oplus M \xrightarrow{-\beta} Y$, apply $\text{Hom}_T(-, V)$ to this triangle, and get the following exact sequence of right $\Lambda_0$-modules:

$$\text{Hom}_T(M_1 \oplus M, V) \to \text{Hom}_T(V, V) \to C \to 0,$$

where $C$ is the cokernel of $\text{Hom}_T(\tilde{\alpha}, V)$. Now, applying $\text{Hom}_{\Lambda_0}^\oplus(\text{Hom}_T(M, V), -)$ to the above exact sequence, we get another exact sequence which is isomorphic to the following exact sequence:

$$\text{Hom}_T(M_1 \oplus M, M) \xrightarrow{(\tilde{\alpha}, M)} \text{Hom}_T(V, M) \to \text{Hom}_{\Lambda_0}^\oplus(\text{Hom}_T(M, V), C) \to 0.$$

Since $\tilde{\alpha}$ is a left add$(M)$-approximation of $V$, the map $\text{Hom}_T(\tilde{\alpha}, M)$ is surjective, and consequently $\text{Hom}_{\Lambda_0}^\oplus(\text{Hom}_T(M, V), C) = 0$. So, the right $\Lambda_0$-module $C$ has no composition factors in $\text{top}(\text{Hom}_T(M, V))$, and that $C$ has composition factors only in $\text{top}(\text{Hom}_T(X, V))$. This is equivalent to saying that the $\Lambda_0$-module $D(C)$ has composition factors only in $\text{soc}(\text{DHom}_T(X, V))$ which is isomorphic to $\text{top}(\text{Hom}_T(V, X))$.

Let $x : V \to V$ be an element in $I' \subseteq \Lambda_0$. Then $x$ factorises through $-\tilde{\omega}[-1]$, or equivalently, $x\tilde{\alpha} = 0$. This implies that $(\text{DHom}_T(x, V))(\text{DHom}_T(\tilde{\alpha}, V)) = 0$. Thus the image of $\text{DHom}_T(x, V)$ is contained in the kernel of $\text{DHom}_T(\tilde{\alpha}, V)$, which is isomorphic to $D(C)$. Therefore, if $\text{DHom}_T(x, V) \neq 0$, then the top of the image of $\text{DHom}_T(x, V)$ is contained in $\text{add}(\text{top}_{\Lambda_0} \text{Hom}_T(V, X)) \cap \text{add}(\text{top}(\Lambda_0) \text{D}\Lambda_0)) = 0$, this
is a contradiction. Thus we must have \( \operatorname{Hom}_F(x, V) = 0 \). Since \( \operatorname{Hom}_F(-, V) \) is a duality from \( \text{add}(V) \) to \( \Lambda_0^{\text{proj}} \)-gproj, we obtain \( x = 0 \). Thus \( I' = 0 \). \( \square \)

**Remark.** If we substitute “\( \text{add}(M) \)-split” for “left \( \text{add}(M) \), \( \Phi \)\)-approximation” and “right \( \text{add}(M) \), \( -\Phi \)\)-approximation” in Proposition 3.9, and if we consider \( E_{F, \Phi}^b(V) \) and \( E_{F, \Phi}^b(W) \) instead of \( \Lambda_0 \) and \( \Gamma_0 \), then Proposition 3.9 is still true. The proof is almost the same.

For the derived category of an abelian category, the following result provides an explicit example for \( I = 0 = J \).

**Proposition 3.10.** Let \( \mathcal{A} \) be an abelian category, and let \( M \) be an object of \( \mathcal{A} \). Suppose that \( 0 \to X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to 0 \) is an exact sequence in \( \mathcal{A} \) with \( M_1 \in \text{add}(M) \). Consider the induced triangle \( X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1] \) in \( \mathcal{D}^b(\mathcal{A}) \). Then the ideals \( I \) and \( J \) defined in Theorem 3.1 vanish.

**Proof.** Every exact sequence \( 0 \to X \to M_1 \to Y \to 0 \) in \( \mathcal{A} \) gives rise to a triangle \( X \to M_1 \to Y \to X[1] \) in \( \mathcal{D}^b(\mathcal{A}) \). Now we show that the exactness of the given sequence in \( \mathcal{A} \) implies that the two ideals \( I \) and \( J \) in Theorem 3.1 are equal to zero. Since \( I \) is contained in \( \operatorname{End}_{\mathcal{D}^b(\mathcal{A})}(X \oplus M) \), it is sufficient to show that if a morphism \( x : X \oplus M \to X \oplus M \) factorises through \( \text{add}(M) \) and \( w[-1] \), then \( x = 0 \). Let \( x \) be such a morphism. Then we see immediately that \( x\alpha = 0 \) in \( \mathcal{D}^b(\mathcal{A}) \). Since \( \mathcal{A} \) is fully embedded in \( \mathcal{D}^b(\mathcal{A}) \), we also have \( x\alpha = 0 \) in \( \mathcal{A} \). Consequently, \( x = 0 \) since \( \alpha \) is injective in \( \mathcal{A} \). Thus \( I = 0 \). Dually, we can show \( J = 0 \). Hence Proposition 3.10 holds true. \( \square \)

As an immediate application of the proof of Theorem 3.1 together with a result on derived equivalences in [20], we have the following corollary.

**Corollary 3.11.** We keep all assumptions of Theorem 3.1. If \( \overline{\mathcal{A}} \) and \( \mathcal{T} \) both are left coherent rings (for example, if \( \Phi \) is finite and \( \mathcal{T} = \mathcal{D}^b(\mathcal{A}) \) with \( \mathcal{A} \) a finite dimensional algebra over a field), then \( \limsup \dim(\overline{\mathcal{A}}) - 1 \leq \dim(\mathcal{T}) \leq \liminf \dim(\overline{\mathcal{A}}) - 1 \), where \( \dim(\overline{\mathcal{A}}) \) stands for the finitistic dimension of \( \overline{\mathcal{A}} \).

Recall that, given a ring \( S \) with identity, the *finitistic dimension* of \( S \) is defined to be the supremum of the projective dimensions of finitely generated \( S \)-modules of finite projective dimension.

Since the map \( q \) in the proof of Theorem 3.1 is not always injective, the tilting complex \( T^\bullet \) is not, in general, isomorphic in \( \mathcal{D}^b(\mathcal{E}_{F, \Phi}^b(V)/I) \) to a tilting module. Thus the derived equivalence presented in Theorem 3.1 is not given by a tilting module in general (in contrast with the situation of Theorem 2.3). In fact, it is easy to see that the derived equivalence in Theorem 3.1 is given by a tilting module if the kernel of \( \mathcal{E}_{F, \Phi}^b(V, \alpha) \) is \( I \cdot \mathcal{E}_{F, \Phi}^b(V, X) \).

Moreover, a small additive category may be embedded into an abelian category of coherent functors (see [18, Chapter IV, Section 2]). This will, however, not in general turn a \( \mathcal{D} \)-split sequence in the additive category into an exact sequence in the abelian category since otherwise the sequence would split, and therefore cannot provide a triangle in the derived category of the abelian category. Consequently, Theorem 2.3 cannot be obtained from Theorem 3.1 by taking \( \Phi = \{0\} \) and embedding an additive category into an abelian category.

Finally, we mention that Theorem 3.1 generalises the result [11, Proposition 5.1] by choosing \( \Phi = \{0\} \). Indeed, under the conditions of [11, Proposition 5.1], the ideals \( I \) and \( J \) in Theorem 3.1 vanish. Theorem 3.1 covers various other situations, some of which will be discussed in the next section.
4 Φ-Yoneda algebras in some explicit situations

In this section, we shall describe some natural habitats for Theorem 3.1 and relate it to several widely used concepts that fit with or simplify the assumptions of Theorem 3.1. Throughout, we choose F to be the shift functor of the triangulated category considered.

We note that Alex Dugas, in independent work [5] that also is motivated by [11], has constructed derived equivalent pairs of symmetric algebras. As explained in [5] (Remark (3) in section 4) his examples appear in our framework, too.

4.1 Derived categories of Artin algebras

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Corollary 4.4. Let $A$ be an Artin algebra, and let $X$ be an $A$-module such that $\text{Ext}^i_A(X, A) = 0$ for all $1 \leq i < n + 2$ with $n$ a fixed positive integer or infinity. Then, for any admissible subset $\Phi$ of $\mathbb{N}_n$, the perforated Yoneda algebras $E^\Phi_A(A \oplus X)$ and $E^\Phi_A(A \oplus \Omega(X))$ are derived equivalent.

Proof. If $\text{Ext}^i_A(X, A) = 0$ for a fixed $i \geq 1$, then $0 \rightarrow \Omega^i(X) \rightarrow P_{i-1} \rightarrow \Omega^{i-1}(X) \rightarrow 0$ is an add$(AA)$-split sequence in $A$-mod, where $P_i$ is a projective cover of $\Omega^i(X)$. Using this fact, Corollary 4.4 follows immediately from Corollary 4.2. □

The condition $\text{Ext}^i_A(X, A) = 0$ on $X$ in Corollary 4.4 is related to the context of the Generalised Nakayama Conjecture. This states that if an $A$-module $T$ satisfies $\text{Ext}^i_A(A \oplus T, A \oplus T) = 0$ for all $i > 0$ then $T$ should be projective. The above Corollary 4.4 (or [11, Theorem 1.1]) describes the shape of the syzygy modules $\Omega^i(X)$: If $X$ is indecomposable and non-projective and satisfies $\text{Ext}^i_A(X, A) = 0$ for all $i > 0$, then, for each $j \geq 0$, there is an indecomposable non-projective module $L_j$ such that $\Omega^j(X) \simeq L_j^{m_j}$ for an integer $m_j > 0$.

In Corollary 4.4, there are isomorphisms $\text{Ext}^i_A(X, X) \simeq \text{Ext}^i_A(\Omega(X), \Omega(X))$ for all $i \geq 1$. Thus the algebras $E^\Phi_A(A \oplus X)$ and $E^\Phi_A(A \oplus \Omega(X))$ are the extensions of $\text{End}_A(A \oplus X)$ and $\text{End}_A(A \oplus \Omega(X))$ by the same ideal $E^{\Phi(0)}_A(X, X)$, respectively. The algebras $E^\Phi_A(X \oplus M)$ and $E^\Phi_A(M \oplus Y)$ in Corollary 4.2, however, are the extensions of $\text{End}_A(X \oplus M)$ and $\text{End}_A(M \oplus Y)$ by possibly different ideals $E^{\Phi(0)}_A(M) \oplus E^{\Phi(0)}_A(X)$ and $E^{\Phi(0)}_A(M) \oplus E^{\Phi(0)}_A(Y)$, respectively.

Recall that a module $M \in A$-mod is called reflexive if the evaluation map

$$\alpha_M : M \rightarrow M^{**} := \text{Hom}_A(\text{Hom}_A(M, A), A)$$

is an isomorphism of modules.

Corollary 4.5. Let $M$ be a reflexive $A$-module. Then, for any subset $0 \in \Phi \subseteq \{0, 1\}$, the perforated Yoneda algebras $E^\Phi_A(D(A) \oplus D\text{Tr}(M))$ and $E^\Phi_A(D(A) \oplus \Omega^{-1}(D\text{Tr}(M)))$ are derived equivalent, where $\Omega^{-1}$ is the co-syzygy operator.

Proof. By [1, IV, Proposition 3.2], the kernel and cokernel of the evaluation map $\alpha_M$ are $\text{Ext}^2_{A^\text{op}}(\text{Tr}(M), A)$ and $\text{Ext}^2_{A^\text{op}}(\text{Tr}(M), A)$, respectively. As $E^\Phi_A(U) \simeq E^\Phi_{A^\text{op}}(D(U))^\text{op}$ for any $A$-module $U$, Corollary 4.5 follows from Corollary 4.4 for right modules. □

A special case of Corollary 4.4, is a result on self-injective algebras that has been obtained in [12, Corollary 3.14]:

Corollary 4.6. If $A$ is a self-injective Artin algebra, then, for any admissible subset $\Phi$ of $\mathbb{N}$, the perforated Yoneda algebras $E^\Phi_A(A \oplus X)$ and $E^\Phi_A(A \oplus \Omega(X))$ are derived equivalent.

Another concept related to the Generalised Nakayama Conjectures and to modules being projective and injective, is the dominant dimension of an algebra or a module.

Suppose that $A$ is an Artin $R$-algebra. By definition, the dominant dimension of $A$ is greater than or equal to $n$ if in the minimal injective resolution of $AA$:

$$0 \rightarrow A \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_{n-1} \rightarrow I_n \rightarrow \cdots,$$

the first $n$ injective $A$-modules $I_0, \cdots, I_{n-1}$ are projective. In this case we write $\text{dom.dim}(A) \geq n$. Let $C_i$ be the i-th cosyzygy of $A$, that is, the cokernel of the map $I_{i-1} \rightarrow I_i$.

For an $A$-module $X$, we define $a(X)$ to be the number of non-isomorphic indecomposable direct summands of $M$. The self-injective measure of $A$ is defined to be the number $m(A) := a(A) - a(I_0)$, where $I_0$ is an injective hull of $A$. Thus, if $A$ is self-injective, then $m(A) = 0$. If $\text{dom.dim}(A) \geq 1$, then $A$ is self-injective if and only if $m(A) = 0$. So the Nakayama conjecture can be reformulated as: If $\text{dom.dim}(A) = \infty$, then $m(A) = 0$. 22
Corollary 4.7. Let $A$ be an Artin algebra, and let $T$ be the direct sum of all non-isomorphic indecomposable projective-injective $A$-modules.

1. If $\text{dom.dim}(A) \geq n \geq 2$, then $\text{End}_A(T \oplus C_i)$ is derived equivalent to $A$ for $1 \leq i < n$. 
2. If $\text{dom.dim}(A) \geq n + 1 < \infty$, then $m(A) = a(C_n)$.

Proof. Since the sequence $0 \to C_{i-1} \to I_i \to C_i \to 0$ is an add$(I_i)$-split sequence (or an add$(T)$-split sequence), the orthogonality conditions in Corollary 4.2 are trivially satisfied. Derived equivalence preserves the number of non-isomorphic simple modules. Therefore, Corollary 4.7 follows now from Corollary 4.2. Here we also use the observation that add$(C_i) \cap \text{add}(I_j) = \{0\}$ for all $0 \leq i, j \leq n$. Alternatively, one can also use Theorem 2.3 to prove this corollary.$\Box$

Examples of algebras of dominant dimension at least $n$ can be obtained in the following way: Let $A$ be a self-injective algebra and $X$ an $A$-module. If $\text{Ext}^i_A(X,A) = 0$ for all $1 \leq i \leq n$, then $\text{dom.dim}(\text{End}_A(A \oplus X)) \geq n + 2$.

Finally, we remark that the condition $\text{Ext}^i_A(X,A) = 0$ for an $A$-module $X$ also appears in Auslander-regular algebras.

Let $\Lambda$ be a $k$-algebra over a field $k$. Recall that $\Lambda$ is called Auslander-regular if $\Lambda$ has finite global dimension and satisfies the Gorenstein condition: if $p < q$ are non-negative integers and $M$ is a finitely generated (left or right) $\Lambda$-module, then $\text{Ext}^q(\Lambda, \Lambda) = 0$ for every submodule $N$ of $\text{Ext}^q(\Lambda, M, \Lambda)$. Here, if $M$ is a right $\Lambda$-module, then $N$ is a left $\Lambda$-module. Let $j(M)$ be the minimal number $r \geq 0$ such that $\text{Ext}^r(\Lambda, \Lambda, \Lambda) \neq 0$. Then for any submodule $N$ of $\text{Ext}^r(\Lambda, M, \Lambda)$, we have $\text{Ext}^r(\Lambda, N, \Lambda) = 0$ for $0 < i < j(M)$. Thus:

Corollary 4.8. Let $\Lambda$ be an Auslander-regular $k$-algebra, and $M$ a finitely generated right $\Lambda$-module. Then, for any submodule $X$ of $\text{Ext}^r(\Lambda, M, \Lambda)$, and any admissible subset $\Phi$ of $\mathbb{N}_{j(M)-2}$, the algebras $E^r(\Lambda \oplus X)$ and $E^r(\Lambda \oplus \Omega(X))$ are derived equivalent.

4.2 Frobenius categories

Let $\mathcal{A}$ be a Frobenius abelian category, that is, $\mathcal{A}$ is an abelian category with enough projective objects and enough injective objects such that the projective objects coincides with the injective objects. We denote by $\mathcal{A}_b$ the stable category of $\mathcal{A}$ modulo projective objects. It is shown in [9] that $\mathcal{A}$ is a triangulated category, in which the shift functor $[1]$ is just the co-syzygy functor $\Omega^{-1}$, and the triangles in $\mathcal{A}_b$ are all induced by short exact sequences in $\mathcal{A}$. For each morphism $f : U \to V$ in $\mathcal{A}$, we denote by $f$ the image of $f$ under the canonical functor from $\mathcal{A}$ to $\mathcal{A}_b$. Note that the objects of $\mathcal{A}_b$ are the same as those of $\mathcal{A}$.

Lemma 4.9. Let $\Phi$ be an admissible subset of $\mathbb{N}$, and let $M$, $X$, and $Y$ be objects in $\mathcal{A}$. Then

1. For arbitrary $0 \neq i \in \mathbb{N}$ and $U, U' \in \mathcal{A}$, there is an isomorphism

$$\text{Hom}_{\mathcal{A}_b}\left(U, U'[i]\right) \simeq \text{Hom}_{\mathcal{A}}\left(U, U'[i]\right),$$

which is functorial in $U$ and $U'$;

2. A monomorphism $\alpha : X \to M_1$ in $\mathcal{A}$ is a left $(\text{add}(M), \Phi)$-approximation of $X$ in $\mathcal{D}^b(\mathcal{A})$ if and only if $\underline{\alpha}$ is a left $(\text{add}(M), \Phi)$-approximation of $X$ in $\mathcal{A}_b$;

3. An epimorphism $\beta : M_2 \to Y$ in $\mathcal{A}$ is a right $(\text{add}(M), -\Phi)$-approximation of $Y$ in $\mathcal{D}^b(\mathcal{A})$ if and only if $\underline{\beta}$ is a right $(\text{add}(M), -\Phi)$-approximation of $Y$ in $\mathcal{A}_b$.

Proof. 1. For $0 \neq i \in \mathbb{N}$, the isomorphisms

$$\text{Hom}_{\mathcal{D}^b(\mathcal{A})}\left(U, U'[i]\right) \simeq \text{Ext}_{\mathcal{A}}^i(U, U') \simeq \text{Hom}_{\mathcal{A}}(U, \Omega^{-i}U') = \text{Hom}_{\mathcal{A}}(U, U'[i]).$$

are functorial in $U$ and $U'$. Thus (1) follows.
(2) First, let \(0 \neq i \in \Phi\). By (1), there is a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{A}^b}(M_1, M[i]) & \xrightarrow{(\alpha, M[i])} & \text{Hom}_{\mathcal{A}^b}(X, M[i]) \\
\sim & & \sim \\
\text{Hom}_{\mathcal{A}}(M_1, M[i]) & \xrightarrow{(\alpha, M[i])} & \text{Hom}_{\mathcal{A}}(X, M[i])
\end{array}
\]

Thus, the map \(\text{Hom}_{\mathcal{A}}(\alpha, M[i])\) is surjective if and only if \(\text{Hom}_{\mathcal{A}^b}(\alpha, M[i])\) is surjective. Now we consider the case \(i = 0\). If every morphism from \(X\) to \(M\) in \(\mathcal{A}\) factorises through \(\alpha\), then every morphism from \(X\) to \(M\) in \(\mathcal{A}\) factorises through \(\alpha\). Conversely, assume that every morphism from \(X\) to \(M\) in \(\mathcal{A}\) factorises through \(\alpha\). Let \(f : X \to M\) be a morphism in \(\mathcal{A}\). Then \(f = \alpha h\) for some \(h : M_1 \to M\) in \(\mathcal{A}\). Thus \(f = \alpha h\) in \(\mathcal{A}\) factorises through a projective object \(P\), say \(f = \alpha h = st\) for some \(s : X \to P\) and \(t : P \to M\) in \(\mathcal{A}\). Since \(P\) is also injective and \(\alpha\) is a monomorphism, there is some morphism \(r : M_1 \to P\) such that \(s = \alpha r\). Altogether, \(f = \alpha h + st = \alpha h + \alpha rt = \alpha(h + rt)\) factorises through \(\alpha\). Thus the statement (2) follows. The proof of (3) is similar to that of (2). □

**Proposition 4.10.** Let \(\Phi\) be an admissible subset of \(\mathbb{N}\). Suppose that \(\mathcal{A}\) is a Frobenius abelian category, that \(M\) is an object in \(\mathcal{A}\), and that \(0 \to X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to 0\) is a short exact sequence in \(\mathcal{A}\) with \(M_1 \in \text{add}(M)\) such that the induced triangle \(X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to X[1]\) in \(\mathcal{A}\) satisfies the conditions in Theorem 3.1. Then the algebras \(\mathcal{E}^\Phi_{\mathcal{A}}(M \oplus Y)\) and \(\mathcal{E}_{\mathcal{A}^b}^\Phi(X \oplus M)\) are derived equivalent.

**Proof.** This follows from Lemma 4.9 and Proposition 3.10. □

### 4.3 Calabi-Yau categories

The theory of Calabi-Yau and cluster categories provides very natural contexts for our construction of derived equivalences.

Let \(k\) be a field, and let \(\mathcal{T}\) be a \(k\)-linear triangulated category which is Hom-finite, that is, the Hom-space \(\text{Hom}_\mathcal{T}(X, Y)\) is finite dimensional over \(k\) for all \(X\) and \(Y\) in \(\mathcal{T}\).

Recall that \(\mathcal{T}\) is called \((n + 1)\)-Calabi-Yau for some non-negative integer \(n\) if there is a natural isomorphism between \(D\text{Hom}_\mathcal{T}(X, Y)\) and \(\text{Hom}_\mathcal{T}(Y, X[n + 1])\) for all \(X\) and \(Y\) in \(\mathcal{T}\), where \(D = \text{Hom}_k(-, k)\) is the usual duality. It follows that \(\mathcal{E}_{\mathcal{T}}^n(M) = \mathcal{E}_{\mathcal{T}}^n(M)\) for \(M \in \mathcal{T}\). (See [14] for more information on Calabi-Yau categories.)

Note that if \(\Phi = \{0, 1, \cdots, n\}\), then \(n - i \in \Phi\) for each \(i \in \Phi\).

**Lemma 4.11.** Let \(\Phi = \{0, 1, \cdots, n\}\). Suppose that \(\mathcal{T}\) is an \((n + 1)\)-Calabi-Yau triangulated category, and that \(M\) is an object in \(\mathcal{T}\). Let \(X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \to X[1]\) be a triangle in \(\mathcal{T}\) with \(M_1 \in \text{add}(M)\). Then:

1. The morphism \(\alpha\) is a left \((\text{add}(M), \Phi)\)-approximation of \(X\) if and only if the morphism \(\beta\) is a right \((\text{add}(M), -\Phi)\)-approximation of \(Y\);
2. If \(\alpha\) is a left \((\text{add}(M), \Phi)\)-approximation of \(X\) and if \(M\) is \(n\)-self-orthogonal, then \(X \in \mathcal{B}^n(M)\cap \mathcal{B}^n(M)\) and \(Y \in \mathcal{B}^n(M)\cap \mathcal{B}^n(M)\).

**Proof.** We will abbreviate \(\text{Hom}_\mathcal{T}(-, -)\) by \((-,-)\). First we assume that \(\alpha\) is a left \((\text{add}(M), \Phi)\)-
approximation of $X$. Now, for each $i \in \Phi$, there is a commutative diagram with exact rows

\[
\begin{array}{c}
(M[-i], M_1) \xrightarrow{(M[-i], \beta)} (M[-i], Y) \\
\cong \quad \cong \\
(M, M_1[i]) \xrightarrow{(M, \beta[i])} (M, Y[i]) \xrightarrow{(M, X[i+1])} (M, M_1[i+1]) \\
D(X, M[n-i]) \xrightarrow{D(M, M[n-i])} D(M_1, M[n-i])
\end{array}
\]

Since $n - i$ is in $\Phi$, and since $\alpha$ is a left (add$(M), \Phi$)-approximation of $X$, the map $(\alpha, M[n-i])$ is surjective, and consequently $D(\alpha, M[n-i])$ is injective. Hence $(M, \alpha[i+1])$ is injective, and therefore $(M[-i], \beta)$ is surjective. This shows that $\beta$ is a right (add$(M), -\Phi$)-approximation of $Y$. The proof of the other implication in (1) can be done similarly.

(2) It follows from (1) and the comment (a) after the proof of Theorem 3.1 that $X \in \mathcal{D}^T_\Phi(M)$ and $Y \in \mathcal{Y}^n(M)$. Since $T$ is $(n+1)$-Calabi-Yau, we have $(M, X[i]) \simeq D(X, M[n+1-i]) = 0$, and $(M, Y[i]) \simeq D(Y, M[n+1-i]) = 0$ for all $0 \neq i \in \Phi$. Thus $X \in \mathcal{D}^T_\Phi(M)$ and $Y \in \mathcal{Y}^n(M)$.

**Corollary 4.12.** Let $\Phi = \{0, 1, \ldots, n\}$, and let $T$ be an $(n+1)$-Calabi-Yau triangulated category. Suppose that $M$ is $n$-self-orthogonal and $Y \in \mathcal{Y}^n(M)$. Let $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ be a triangle in $T$ with $\beta$ a right add$(M)$-approximation of $Y$. Then the algebras $E^T_D(M \oplus X)/I$ and $E^T_D(M \oplus Y)/J$ are derived equivalent, where $I$ and $J$ are defined as in Theorem 3.1.

**Proof.** Since $Y \in \mathcal{Y}^n(M)$, for each $0 \neq i \in \Phi$, the map $(M[-i], M_1) \rightarrow (M[-i], Y) = 0$ induced by $\beta$ is surjective. Taking into account that $\beta$ is a right add$(M)$-approximation of $Y$, we see that $\beta$ is, in fact, a right (add$(M), -\Phi$)-approximation of $Y$. By Proposition 4.11 (1), the map $\alpha$ is a left (add$(M), \Phi$)-approximation of $X$. Since $M$ is $n$-self-orthogonal, the proof can be finished by applying Proposition 4.11 (2) and Corollary 3.7 to the triangle. □

Corollary 4.12 is related to mutations in a Calabi-Yau category. Here are some definitions from [13].

Let $T$ be an $(n+1)$-Calabi-Yau category. An object $T$ in $T$ is called an $n$-cluster tilting object if $T$ is $n$-self-orthogonal, and if any $X \in T$ with Ext$^i_T(T, X) = 0$ for $1 \leq i \leq n$ is in add$(T)$. The object $T$ is called basic if the multiplicity of each indecomposable direct summand of $T$ is one.

Let $T$ be an $n$-cluster basic tilting object in an $(n+1)$-Calabi-Yau category $T$, and $Y$ a direct summand of $T$, that is, $T = Y \oplus M$. Let $\beta : M_1 \rightarrow Y$ be a minimal right add$(M)$-approximation of $Y$, and let

\[ X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \rightarrow X[1] \]

be a triangle containing $\beta$. Note that we allow $Y$ to be decomposable, and that $X$ is indecomposable if and only if $Y$ is indecomposable. The object $X \oplus M$ is called the left mutation of $T$ at $Y$. In the case of tilting modules, $X$ is called a tilting complement to $M$ in the literature (see, for example, [10]). It was pointed out in [13] that the left mutation of $T$ at $Y$ is again an $n$-cluster tilting object (for some special cases, see [3, 8], and also [17, p.314]). In fact, this can be seen in the following way: The proof of Corollary 4.12 and (comment) on the conditions of Theorem 3.1 imply that $T' := M \oplus X$ is $n$-self-orthogonal. Moreover, let $X' \in \mathcal{D}^n(T')$ and consider a triangle $X' \xrightarrow{\alpha'} M' \rightarrow Y' \rightarrow X'[1]$ with $\alpha'$ a left add$(M)$-approximation of $X'$. Then $Y' \in \mathcal{D}^n(T')$ by Lemma 4.11 and the comment (b). Thus $Y' \in \text{add}(T)$, $X' \in \text{add}(T')$, and $T' := X \oplus M$ is again an $n$-cluster tilting object in $T$. The notion of a right mutation of $T$ at $Y$ is dual.

Usually, $\text{End}_T(X \oplus M)$ and $\text{End}_T(M \oplus Y)$ are not derived equivalent. When they are derived equivalent may be an interesting question. Here is a sufficient condition.
Corollary 4.13. Let $\Lambda := \text{End}_F(X \oplus M)$ and $\Gamma := \text{End}_F(M \oplus Y)$. Then

1. $\text{End}_F(X \oplus M)/I$ and $\text{End}_F(M \oplus Y)/J$ are derived equivalent.

2. Suppose that $Y$ is indecomposable. Let $S_X$ be the simple $\Lambda$-module corresponding to $X$, and let $S_Y$ be the simple $\Gamma$-module corresponding to $Y$. Suppose that $S_Y$ is not a submodule of $\Gamma$, and $S_X$ is not a quotient of $D(\Lambda)$. Then $\Lambda$ and $\Gamma$ are derived equivalent.

Proof. Statement (1) is a direct consequence of Corollary 4.12, and (2) follows from (1) and Proposition 3.9. □

Remark. Consider a 2-Calabi-Yau category, and assume that $\text{Ext}_F^1(S_Y, S_Y) = 0$. Then we re-obtain the result [15, Theorem 5.3] from Corollary 4.13 (2).

5 Examples

First, we present an explicit example which satisfies all conditions in Theorem 3.1.

Example 1. Let $k$ be an algebraically closed field of characteristic 2, and let $A := kA_4$ be the group algebra of the alternating group $A_4$. Then there are three simple $A$-modules, which are denoted $k, \omega$, and $\bar{\omega}$, respectively. Their projective covers are $P(k)$, $P(\omega)$ and $P(\bar{\omega})$, respectively. It was shown in [6, V2.4.1, p.129] that $kA_4$ is Morita equivalent to the following algebra given by quiver

\[
\begin{array}{c}
\omega \\
\downarrow \beta_2 \\
\downarrow \beta_1 \\
k \\
\end{array}
\begin{array}{c}
\alpha_1 \\
\downarrow \alpha_3 \\
\downarrow \alpha_2 \\
\end{array}
\]

and relations $\alpha_i \beta_{i+1} - \beta_i \alpha_{i+1} = \alpha_i \alpha_{i+1} = \beta_i \beta_{i-1} = 0$, where the subscripts are considered modulo 3.

As this algebra is symmetric, the Auslander-Reiten translation $D\text{Tr}$ is just the second syzygy $\Omega^2$. Thus a direct computation shows that the Auslander-Reiten quiver of this algebra has a component of the following form:

\[
\begin{array}{c}
\cdots \Omega^2(k) \\
\cdots \Omega(k) \\
\cdots \Omega^2(\omega) \\
\cdots \Omega^2(k) \\
\end{array}
\begin{array}{c}
\cdots \Omega(\omega) \\
\cdots \Omega(k) \\
\cdots \Omega(\omega) \\
\cdots \Omega(k) \\
\end{array}
\begin{array}{c}
P(\omega) \\
\Omega^{-1}(\omega) \\
P(\omega) \\
\Omega^{-1}(\omega) \\
\end{array}
\begin{array}{c}
\Omega^{-2}(\omega) \\
\Omega^{-3}(\omega) \\
\Omega^{-2}(\omega) \\
\Omega^{-3}(\omega) \\
\end{array}
\begin{array}{c}
\cdots \Omega^2(k) \\
\cdots \Omega^2(\omega) \\
\cdots \Omega^2(k) \\
\end{array}
\]

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Consider the Auslander-Reiten sequence
\[ 0 \rightarrow \Omega^3(\omega) \rightarrow \Omega^2(k) \oplus \Omega^2(\omega) \rightarrow \Omega(\omega) \rightarrow 0. \]
Let \( X = \Omega^3(\omega), Y = \Omega(\omega), \) and \( M = \Omega^2(k) \oplus \Omega^2(\omega). \) This sequence provides an Auslander-Reiten triangle in the triangulated category \( \text{A-mod}: \)
\[ X \rightarrow M \rightarrow Y \rightarrow X[1]. \]
We shall check that this triangle satisfies the conditions of Theorem 3.1.

We choose \( \Phi = \{0, 1\} \) and \( F = [1]. \) Since this is an Auslander-Reiten triangle in \( \text{A-mod}, \) the map \( X \rightarrow M \) is a left \((\text{add}(M), \Phi)\)-approximation of \( X, \) and the map \( M \rightarrow Y \) is a right \((\text{add}(M), -\Phi)\)-approximation of \( Y \) (see the example at the end of Section 2). It follows from the above Auslander-Reiten quiver of \( A \) that \( \text{Ext}^1_A(M, X) \simeq \text{Hom}_A(M, \Omega^{-1}(X)) \simeq \text{Hom}_A(\Omega^2(k) \oplus \Omega^2(\omega), \Omega^2(\omega)) = 0 \) and \( \text{Ext}^1_A(Y, M) \simeq \text{Hom}_A(Y, \Omega^{-1}(M)) = \text{Hom}_A(\Omega(\omega), \Omega(k) \oplus \Omega(\omega)) = 0. \) Thus the above triangle in \( \text{A-mod} \) satisfies all conditions in Theorem 3.1, and therefore, by Proposition 4.10, the algebras \( E^\Phi_A(M \oplus X) \) and \( E^\Phi_A(M \oplus Y) \) are derived equivalent.

Furthermore, we have \( \text{Ext}^i_A(M, M) \simeq \text{Hom}_A(M, \Omega^{-i}(M)) = \text{Hom}_A(\Omega(\omega), \Omega(k) \oplus \Omega(\omega)). \) There is an epimorphism from \( \Omega(k) \) to \( \omega \) and an epimorphism from \( \Omega(\omega) \) to \( k. \) The latter cannot factorise through a projective module, so we get \( \dim \text{Ext}^1_A(M, M) = 2. \) Moreover, there is an epimorphism from \( \Omega(k) \) to \( \omega \) and an epimorphism from \( \Omega(\omega) \) to \( \omega. \) This implies \( \dim \text{Ext}^1_A(M, M) = 2. \) Similarly, \( \dim \text{Ext}^1_A(Y, M) = 2. \) Note that all indecomposable modules appearing in the Auslander-Reiten triangle are \( 1 \)-self-orthogonal. A more precise calculation shows that \( \dim \text{Ext}^1_A(M \oplus X) = 33 \) and \( \dim \text{Ext}^1_A(M \oplus Y) = 21. \)

The following example shows that the Ext-orthogonality conditions in Corollary 4.2 and therefore in Theorem 3.1 cannot be dropped.

**Example 2.** Let \( A \) be the algebra (over a field \( k \)) given by the following quiver with relations:

\[
\begin{array}{c}
\bullet \\
1 \\
\downarrow \beta \\
\bullet \\
2
\end{array}
\]

\( \alpha \cdot 2 = 0 = \alpha \beta. \)

This example is in a class of examples constructed by Small [24]. The algebra \( A \) is of finite representation type, its finitistic dimension equals one, while the finitistic dimension of the opposite algebra \( A^{op} \) is zero.

We denote by \( S(i) \) and \( P(i) \) the simple and projective modules corresponding to the vertex \( i, \) respectively. Let \( M_i \) be the quotient module of \( P(2) \) by \( S(i), \) and \( M := M_1 \oplus M_2 = D(A_A), \) where \( D \) is the usual duality. Then there is an Auslander-Reiten sequence
\[ 0 \rightarrow X := P(2) \rightarrow M \rightarrow S(2) =: Y \rightarrow 0. \]
This is an \( \text{add}(M) \)-split sequence in \( \text{A-mod}. \)

If we take \( \Phi = \{0, 1\}, \) then \( E^\Phi_A(X \oplus M) = \text{End}_A(X \oplus M). \) An easy calculation shows that \( \text{End}_A(X \oplus M) \) is a quasi-hereditary algebra, and thus has finite global dimension. The algebra \( E^\Phi_A(M \oplus Y) \) contains a loop which is given by the short exact sequence induced by the loop \( \alpha \) at the vertex 2. Thus it has infinite global dimension by [16]. It follows that \( E^\Phi_A(X \oplus M) \) and \( E^\Phi_A(M \oplus Y) \) cannot be derived equivalent since derived equivalences preserve the finiteness of global dimensions. Also, one can see that \( \text{Ext}^i_A(X, M) = 0 = \text{Ext}^i_A(M, M) \) and \( \text{Ext}^i_A(Y, M) \neq 0 = \text{Ext}^i_A(M, M) \) for \( i \geq 1. \) This example shows that the orthogonality conditions in Corollary 4.2 cannot be omitted. Moreover, it shows that the result in [11, Theorem 1.1] cannot be extended from endomorphism algebras to \( \Phi \)-Yoneda algebras without any additional conditions.
A two functors version of Theorem 1.1

In Theorem 3.1, there is only one functor $F$ involved. When working with the derived category of a hereditary algebra, or the stable category of a self-injective algebra, or the derived category of coherent sheaves of a projective variety over $\mathbb{C}$, apart from the shift functor there are other prominent functors, for example, the Auslander-Reiten translation $\text{DTr}$. To have available a general statement of construction of derived equivalences, which is similar to Theorem 3.1, we define $\Phi$-perforated Yoneda algebras for two functors over a triangulated category, and formulate a two-functor version of Theorem 3.1. In this appendix, we summarise the ingredients for a generalisation of Theorem 3.1. The proof of this generalisation is analogous to that of Theorem 3.1, but more technical and tedious. So we omit it here.

Let $\Phi$ be a subset of $\mathbb{N} \times \mathbb{N}$ which we consider as a semigroup with ordinary addition. Let $\mathcal{T}$ be a triangulated $R$-category with shift functor $[1]$, and let $X$ be an object in $\mathcal{T}$.

Suppose that $F$ and $G$ are two triangle functors from $\mathcal{T}$ to itself, such that $FG$ is naturally isomorphic to $GF$. For $X$ in $\mathcal{T}$, let $\delta(i,j,X) : F^iG^jX \to G^iF^jX$ be an isomorphism induced from the natural transformation $FG \sim GF$. Then we define

$$E^{F,G}_{\mathcal{T}}(X) := \bigoplus_{(i,j) \in \Phi} \text{Hom}_{\mathcal{T}}(X, G^iF^jX),$$

with elements of the form $(f_{i,j})(i,j)_{(i,j) \in \Phi}$, where $f_{i,j} : X \to G^iF^jX$. The multiplication on $E^{F,G}_{\mathcal{T}}(X)$ is given by

$$(f_{i,j})(i,j)_{(i,j) \in \Phi} \cdot (g_{i,j})(i,j)_{(i,j) \in \Phi} = \left( \sum_{(p,q) \in \Phi} \sum_{(r,s) \in \Phi} f_{i+p,j+q}(G^pF^qg_{p,q})(G^sF^r\delta(p,v,F^qX)) \right)_{(l,m) \in \Phi}.$$

A general model for the above definition is: Given a bi-graded algebra $\Lambda = \bigoplus_{(i,j) \in \mathbb{Z}} \Lambda_{i,j}$, we define $\Lambda(\Phi) = \bigoplus_{(i,j) \in \Phi} \Lambda_{i,j}$, and a multiplication by $a_{i,j} \cdot a_{p,q} = a_{i,j}a_{p,q}$ if $(i+p,j+q) \in \Phi$, and zero otherwise.

If $\Phi$ is admissible, for example, $\Phi$ is the cartesian product of two admissible sets in $\mathbb{Z}$, then $\Lambda(\Phi)$ is an associative algebra. So, we have to check that, given two auto-isomorphism functors $F$ and $G$ on $\mathcal{T}$, the $R$-module $E^{F,G}_{\mathcal{T}}(X) := \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}(X, G^iF^jX)$ is an associative algebra with respect to the above multiplication. This can be based on the following lemma.

**Lemma A.1.** Suppose $F$ and $G$ are two triangle functors from $\mathcal{T}$ to itself such that $FG$ is naturally isomorphic to $GF$. For any triangle functor $L$ from $\mathcal{T}$ to itself, there is a natural isomorphism $\delta(i,j,L) : F^iG^jL \to G^iF^jL$ for all $i,j \geq 0$ such that, for $p,q,r,s \in \mathbb{N}$,

1. $\delta(p+q,r,L) = \delta(p,r,G^qL)(G^p\delta(q,r,L))$;
2. $\delta(p,r+s,L) = (F^s\delta(p,s,F^rL))\delta(p,r,F^sL)$.

**Proof.** For functors $L_1$ and $L_2$ from $\mathcal{T}$ to itself, we define $L_1\delta(1,1,L_2) : L_1FGL_2 \to L_1GFL_2$ to be the induced natural isomorphism from the functor $L_1FGL_2$ to the functor $L_1GFL_2$. So, $\delta(1,1,1_{\mathcal{T}})$ is just the given natural isomorphism from $FG$ to $GF$. Now we shall construct inductively a natural isomorphism $\delta(i,j,L)$ from $F^iG^jL$ to $G^iF^jL$ for all non-negative integers $i$ and $j$ and functors $L$ from $\mathcal{T}$ to itself.

If $i=0$ or $j=0$, then $F^iG^jL = G^iF^jL$, and we define $\delta(i,j,L)$ to be the identity natural transformation. For each positive integer $j>1$, we assume that $\delta(1,j-1,L)$ is defined. Now we define

$$\delta(1,j,L) := (F\delta(1,j-1,L))\delta(1,1,F^{j-1}L).$$
For each positive integer $i > 1$, assume that $\delta(i - 1, j, L)$ is defined. We define

$$\delta(i, j, L) := \delta(1, j, G^{i-1}L)(G\delta(i - 1, j, L)).$$

(1) It is straightforward to check that (1) holds for $p + q \leq 2$. We shall prove (1) by induction on $p + q$. Now assume that $p + q > 2$. Then we have

$$\delta(p + q, r, L) = \delta(1, r, G^{p+q-1}L)(G\delta(p + q - 1, r, L)) \quad \text{(by definition)}$$

$$= \delta(1, r, G^{p+q-1}L)G\left(\delta(p - 1, r, G^qL)(G^{p-1}\delta(q, r, L))\right) \quad \text{(by induction)}$$

$$= \left(\delta(1, r, G^{p+q-1}L)(G\delta(p - 1, r, G^qL))\right)(G^p\delta(q, r, L))$$

$$= \delta(p, r, G^qL)(G^p\delta(q, r, L)) \quad \text{(by definition).}$$

This proves (1).

(2) We first prove (2) for $p = 0, 1$. If $p = 0$, then (2) is clearly true. Now suppose $p = 1$. We shall show (2) by induction on $r + s$. In fact, if $r + s \leq 2$, it is straightforward to check (2). Now we assume that $r + s > 2$. Then we have

$$\delta(1, r + s, L) = (F\delta(1, r + s - 1, L))\delta(1, 1, F^{r+s-1}L) \quad \text{(by definition)}$$

$$= F\left((F^{r-1}\delta(1, r, L))\delta(1, s - 1, F^sL)\right)\delta(1, 1, F^{r+s-1}L) \quad \text{(by induction)}$$

$$= (F^s\delta(1, r, L))(\delta(1, 1, F^{r+s-1}L))\delta(1, 1, 1)$$

$$= (F^s\delta(1, r, L))\delta(1, s, F^sL) \quad \text{(by definition).}$$

This proves (2) for $p = 1$. Now assume $p > 1$. Then

$$\delta(p, r + s, L) = \delta(1, r + s, G^{p-1}L)(G\delta(p - 1, r + s, L)) \quad \text{(by definition)}$$

$$= (F^s\delta(1, r, G^{p-1}L))\delta(1, s, F^sG^{p-1}L)G\left(\delta(p - 1, r, F^sL)\delta(1, 1, F^{p-1}L)\right)(G\delta(p - 1, s, F^sL)).$$

Since $\delta(1, s, F^sG^{p-1}L)$ is a natural transformation from $F^sGF^sG^{p-1}L$ to $GF^sF^sG^{p-1}L$, the following diagram of natural transformations is commutative:

$$\begin{array}{ccc}
F^sGF^sG^{p-1}L & \xrightarrow{\delta(1, s, F^sG^{p-1}L)} & GF^sF^sG^{p-1}L \\
F^sG\delta(p - 1, r, L) & \xrightarrow{\delta(1, s, F^sG^{p-1}L)} & GF^sG(p - 1, s, F^sL)
\end{array}$$

Hence

$$\delta(p, r + s, L) = (F^s\delta(1, r, G^{p-1}L))\left(\delta(1, s, F^sG^{p-1}L)(GF^s\delta(p - 1, r, L))\right)(G\delta(p - 1, s, F^sL)).$$

$$= (F^s\delta(1, r, G^{p-1}L))\left(F^sG\delta(p - 1, r, L))\delta(1, s, G^{p-1}F^sL)\right)(G\delta(p - 1, s, F^sL)).$$

$$= F^s\left(\delta(1, r, G^{p-1}L)(G\delta(p - 1, r, L))\right)(\delta(1, s, G^{p-1}F^sL)(G\delta(p - 1, s, F^sL))).$$

$$= (F^s\delta(p, r, L))\delta(p, s, F^sL)$$

This proves (2). \(\square\)

Remark. If, in addition, $F$ and $G$ are auto-isomorphisms, then Lemma A.1 remains valid for $i, j, p, q, r$ and $s$ any integers.

Let $D$ be a full subcategory of $T$, and $X$ an object of $T$. A morphism $f : X \to D$ with $D \in D$ is called a left $(D, F, G, \Phi)$-approximation of $X$ if Hom$_T(f, G^iF^jD') :$ Hom$_T(D, G^iF^jD') \to$ Hom$_T(X, G^iF^jD')$
is surjective for every object $D' \in \mathcal{D}$ and $(i, j) \in \Phi$. Dually, we define the right $(\mathcal{D}, F, G, \Phi)$-approximation of $X$.

Given a triangle $0 \to X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ in $\mathcal{T}$ with $M_1 \in \text{add}(M)$ for a fixed $M \in \mathcal{T}$, we define $\tilde{w}[-1] = (-w[-1], 0) : Y[-1] \to X \oplus M$, $\tilde{w} = (0, w)^T$, where $(0, w)^T$ stands for the transpose of the matrix $(0, w)$, and

$$I := \{x = (x_{i,j}) \in E_{t}^{F,G,\Phi}(X \oplus M) \mid x_{i,j} = 0 \text{ for } (0, 0) \neq (i, j) \in \Phi, \text{ and } x_{0,0} \text{ factors through } \text{add}(M) \text{ and } \tilde{w}[-1]\},$$

$$J := \{y = (y_{i,j}) \in E_{t}^{F,G,\Phi}(M \oplus Y) \mid y_{i,j} = 0 \text{ for } (0, 0) \neq (i, j) \in \Phi, \text{ and } y_{0,0} \text{ factors through } \text{add}(M) \text{ and } \tilde{w}\}.$$ 

Now, with a proof similar to Theorem 3.1, one can get the following result with two functors.

**Theorem A.2.** Let $\Phi$ be an admissible subset of $\mathbb{Z} \times \mathbb{Z}$, and let $\mathcal{T}$ be a triangulated $R$-category, and let $M$ be an object in $\mathcal{T}$. Assume that there are two triangle auto-isomorphisms $F$ and $G$ from $\mathcal{T}$ to itself such that $FG$ is naturally isomorphic to $GF$ by $\delta : FG \to GF$. Suppose that $X \xrightarrow{\alpha} M_1 \xrightarrow{\beta} Y \xrightarrow{w} X[1]$ is a triangle in $\mathcal{T}$ such that $\alpha$ is a left $(\text{add}(M), F, G, \Phi)$-approximation of $X$ and $\beta$ is a right $(\text{add}(M), F, G, -(\Phi))$-approximation of $Y$. If $\operatorname{Hom}_\mathcal{T}(M, G^*F/X) = 0 = \operatorname{Hom}_\mathcal{T}(Y, G^*F/(M))$ for $(0, 0) \neq (i, j) \in \Phi$, then $E_{t}^{F,G,\Phi}(X \oplus M)/I$ and $E_{t}^{F,G,\Phi}(M \oplus Y)/J$ are derived equivalent.

Taking $G = [1]$ and $F = \text{id}$ in a derived module category yields a result on Ext-algebras. Taking $G = \text{id}$, we recover Theorem 3.1 for the case of $F$ being an arbitrary auto-isomorphism.

**Outline of the proof of Theorem A.2:** Clearly, as in the proof of Lemma 3.3, we can use Lemma 3.2 to show that $I$ and $J$ are ideals in $E_{t}^{F,G,\Phi}(X \oplus M)$ and $E_{t}^{F,G,\Phi}(M \oplus Y)$, respectively. The next step is to check the complex $T^*$, which can be defined analogously as in Lemma 3.5, is a tilting complex. Finally, one needs to prove the isomorphism described in Lemma 3.6. However, this is a word by word verification by following the proof of Lemma 3.6.

**References**


Wei Hu, School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, 100875 Beijing, People’s Republic of China
   Email: huwei@bnu.edu.cn
   Homepage: http://math.bnu.edu.cn/~huwei/

Steffen Koenig, Institut für Algebra and Zahlentheorie, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany
   Email: skoenig@mathematik.uni-stuttgart.de
   Homepage: http://www.iaz.uni-stuttgart.de/LstAGeoAlg/Koenig/

Changchang Xi (Corr. Author), School of Mathematical Sciences, Capital Normal University, 100048 Beijing, People’s Republic of China
   Email: xicc@bnu.edu.cn
   Homepage: http://math.bnu.edu.cn/~ccxi/