On Constructing Ideals of the Hall Algebra of Type $B$*

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Abstract. Let $H_v(A_n)$ and $H_v(B_n)$ be the Hall algebras over $\mathbb{Q}(v)$ of the Dynkin quivers $A_n$ and $B_n$ ($n \geq 1$), respectively, where $v$ is an indeterminate and the quivers have linear orientation. By comparing the quantum Serre relations, we find a natural algebra epimorphism $\pi : H_v(B_n) \rightarrow H_v^2(A_n)$. We determine the kernel of $\pi$ by giving two sets of generators. Let $\varphi$ be the natural algebra homomorphism from $H_v(A_n)$ to the quantized Schur algebra $S_v(n + 1, r)$ ($r \geq 1$) and write $\tilde{\varphi} : H_v^2(A_n) \rightarrow S_v(n + 1, r)$ for the induced map. We obtain several ideals of $H_v(B_n)$ by lifting the kernel of $\varphi$ to the kernel of the composition map $\tilde{\varphi} \circ \pi : H_v(B_n) \rightarrow S_v(n + 1, r)$.

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1 Introduction

Let $H_v(A_n)$ and $H_v(B_n)$ be the Hall algebras over $\mathbb{Q}(v)$ of the Dynkin quivers $A_n$ and $B_n$ ($n \geq 1$), respectively, where $v$ is an indeterminate and the quivers have linear orientation as follows:

$A_n$:

1 $\rightarrow \cdots \rightarrow n - 1 \rightarrow n$

$B_n$:

1 $\rightarrow \cdots \rightarrow (2,1) \rightarrow n - 1 \rightarrow n$

By Ringel [6], the Hall algebras $H_v(A_n)$ and $H_v(B_n)$ are isomorphic to the positive parts of the corresponding quantum groups, and can be described by quantum Serre relations.

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Our main results are the following. By comparing the quantum Serre relations, we find a natural algebra morphism $\pi : H_v(B_n) \to H_{v^2}(A_n)$. We determine the kernel of $\pi$ as an ideal of $H_v(B_n)$ by giving two sets of generators (Theorem 2.3). Let $\varphi$ be the algebra homomorphism from $H_v(A_n)$ to the quantized Schur algebra $S_v(n + 1, r)$ ($r \in \mathbb{N}$) defined in [4]. Write $\tilde{\varphi} : H_{v^2}(A_n) \to S_v(n + 1, r)$ for the $\mathbb{Q}(v)$-algebra map naturally induced by $\varphi$. Let $\psi : H_v(B_n) \to S_v(n + 1, r)$ be the composition map of $\varphi$ and $\pi$. We express the kernel of $\psi$ as the sum of two ideals $I_2(B_n)$ and $\text{Ker}(\pi)$ of $H_v(B_n)$, and also as the direct sum of two subspaces $I_1(B_n)$ and $\text{Ker}(\pi)$ (Theorem 2.5). The $\mathbb{Q}(v)$-bases of $I_2(B_n)$ and $I_1(B_n)$, which are of PBW-type, are obtained in Theorems 2.4 and 2.5, respectively.

It was first explored in [1] that the quantized Schur algebra $S_v(n + 1, r)$ ($r \geq 1$) is closely related to the quantum group of type $A_n$, and hence to the Hall algebra of type $A_n$. In [4] Green determined the kernel of $\varphi : H_v(A_n) \to S_v(n + 1, r)$ explicitly, which has a beautiful basis of PBW-type. Our motivation is to generalize this basis to type $B$ and construct ideals of the Hall algebra $H_v(B_n)$ so that they have representation meaning. It would also be interesting to define the map from $H_v(A_n)$ to the quantized Schur algebra of type $B$ and determine its kernel.

The paper is organized as follows. Section 2 recalls the definition of the Hall algebras $H_v(A_n)$ and $H_v(B_n)$, and states the main theorems. Section 3 proves a so-called generalized quantum Serre relation for type $A_n$. Section 4 proves our theorems using the results developed in Section 3.

We write $\mathbb{N}$ for the set of positive integers, and $\mathbb{N}_0$ for the set of non-negative integers.

2 The Hall Algebras $H_v(A_n)$ and $H_v(B_n)$

- Root systems and Euler forms: Let $\Phi^+(A_n)$ and $\Phi^+(B_n)$ be the sets of positive roots of the simple Lie algebras of type $A_n$ and $B_n$ ($n \geq 1$), respectively. By Gabriel [3], they are in bijections with the sets of isomorphism classes of the indecomposable representations of the quivers $A_n$ and $B_n$, respectively. For a positive root $\alpha$, write $M_\alpha$ for an indecomposable representation corresponding to $\alpha$. We have the following known facts:

1. (Ringel [7]) Write $\Phi^+$ for both $\Phi^+(A_n)$ and $\Phi^+(B_n)$. There exists a ‘good’ order on $\Phi^+$ such that $\Phi^+ = \{\beta_1, \beta_2, \ldots, \beta_N\}$ with $\text{Hom}(M_{\beta_i}, M_{\beta_j}) = 0$ unless $i \leq j$, and $\text{Ext}(M_{\beta_i}, M_{\beta_j}) = 0$ unless $j < i$. We define $\beta_i \succeq \beta_j$ if and only if $i \leq j$, and $\beta_i \prec \beta_j$ if and only if $i < j$.

2. $\# \Phi^+(A_n) = \frac{n(n+1)}{2}$ and $\# \Phi^+(B_n) = n^2$. Identifying the simple roots provides an embedding of $\Phi^+(A_n)$ into $\Phi^+(B_n)$, which is compatible with their ‘good’ orders. Let us denote by $\Phi^+_1$ the subset of $\Phi^+(B_n)$ identified with $\Phi^+(A_n)$, and $\Phi^+_2$ its complement, i.e., $\Phi^+(B_n) = \Phi^+_1 \cup \Phi^+_2$.

3. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots of both $A_n$ and $B_n$. Write $0^{a_1}1^{b_0}c$ for the root $\alpha_{a_1} + \alpha_{a_2} + \cdots + \alpha_{a+b}$ in $\Phi^+(A_n)$ and $\Phi^+_1$, where $b > 0$ and $a + b + c = n$. The roots in $\Phi^+_2$ have the form

$$0^{2a}1^{2b}c = a_{a+1} + \cdots + a_{a+b} + 2a_{a+b+1} + \cdots + 2a_{a+b+c},$$

where $b, c > 0$ and $a + b + c = n$. 

4. ( Crawley-Boevey [2] ) By definition, the Euler form $\langle -, - \rangle$ in type $A_n$ is given by

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } j - i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and that in type $B_n$ is given by

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j \in \{1, \ldots, n-1\}, \\ 1 & \text{if } i = j = n, \\ -2 & \text{if } j - i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

5. ’Good’ order on $\Phi^+(A_n)$ and $\Phi^+_{\ell}$. For $\beta = 0^{a_1}1^{b_1}0^{c_1}$ and $\beta' = 0^{a_2}1^{b_2}0^{c_2}$, $\beta < \beta'$ if and only if either $a_1 > a_2$, or $a_1 = a_2$ and $b_1 > b_2$.

- Hall algebras and PBW-type bases: Let $Q$ be the linearly oriented quiver $A_n$ or $B_n$ with vertices $\{1, 2, \ldots, n\}$. Write $S_i$ for the irreducible representation of $Q$ supported at the vertex $i$ ($1 \leq i \leq n$). Let $\Phi^+ = \{\beta_1, \beta_2, \ldots, \beta_N\}$ be the set of positive roots with respect to the ’good’ order. Let $\mathcal{P}$ be the set of isomorphism classes of finite dimensional $\mathbb{F}_q$-representations of $Q$, where $\mathbb{F}_q$ is a finite field of $q$ elements.

Fix any $[M], [N], [X] \in \mathcal{P}$. By Ringel [5], the Hall number $g_{[M],[N]}^{[X]}$ is a polynomial in $\mathbb{Z}[v]$. When valued at $v = \sqrt{\delta}$, it counts the number of submodules $X_1$ of $X$ satisfying $X_1 \cong N$ and $X/X_1 \cong M$. The Hall algebra of type $Q$, denoted by $H_v(Q)$, is the $\mathbb{Q}(v)$-algebra with basis $\{ [M] : [M] \in \mathcal{P} \}$ and product

$$[M] \ast [N] = v^{\dim(M) \dim(N)} [M] \circ [N],$$

where $[M], [N] \in \mathcal{P}$ and

$$[M] \circ [N] = \sum_{[X] \in \mathcal{P}} g_{[M],[N]}^{[X]} [X].$$

They are called the star product and the diamond product, respectively.

Set $\langle M_\alpha \rangle = v^{\epsilon(\alpha)} [M_\alpha]$ for $\alpha \in \Phi^+$, where $\epsilon(\alpha) = \dim(\text{End}(M_\alpha)) - \dim(M_\alpha)$. Since indecomposable representations have no self-extensions, we have that $\epsilon(\alpha) = \langle \alpha, \alpha \rangle - \dim(M_\alpha)$. By Ringel [6], the Hall algebra $H_v(Q)$ has a PBW-type basis over $\mathbb{Q}(v)$:

$$\{ \langle M_{\beta_1} \rangle^{n_1} \ast \langle M_{\beta_2} \rangle^{n_2} \ast \cdots \ast \langle M_{\beta_N} \rangle^{n_N} : n_1, \ldots, n_N \in \mathbb{N}_0 \},$$

where the divided power

$$\langle M_{\beta} \rangle^{(n)} = \frac{\langle M_{\beta} \rangle^{n}}{[n]_{\beta_{\beta'}}}$$

and $[n]_m! = \prod_{k=1}^n [k]_m$, $[k]_m = \frac{(q^m-q^k)}{(q^m-q^k)} \in \mathbb{Q}(v)$ for $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Furthermore, $H_v(Q)$ is isomorphic to the positive part of the corresponding quantum
group. Hence, the Hall algebra can be described by quantum Serre relations as follows, where the multiplication corresponds to the star product.

The Hall algebra $H_v(A_n)$ is the associative $\mathbb{Q}(v)$-algebra with generators $\{E_i = (S_i) : i = 1, 2, \ldots, n\}$ and relations

\begin{align*}
(A1) & \quad E_i^2 - (v + v^{-1})E_iE_i + E_iE_i = 0 \text{ for } |i - j| = 1, \\
(A2) & \quad E_iE_j = E_jE_i \text{ for } |i - j| > 1.
\end{align*}

Define the root vector $E_\alpha = \langle M_\alpha \rangle$ for $\alpha \in \Phi^+(A_n)$. In particular, $E_\alpha = E_i$ for $1 \leq i \leq n$. Since all $\langle \alpha, \alpha \rangle = 1$, the divided powers are given by $E^{(n\alpha)} = \frac{E^n_\alpha}{\alpha^{n\alpha}}$.

The Hall algebra $H_v(B_n)$ is the associative $\mathbb{Q}(v)$-algebra with generators $\{E_i = (S_i) : i = 1, 2, \ldots, n\}$ and relations

\begin{align*}
(B1) & \quad E_i^2E_j - (v^2 + v^{-2})E_iE_jE_i + E_jE_i^2 = 0 \text{ for } |i - j| = 1 \text{ and } i \neq n, \\
(B2) & \quad E_n^3E_{n-1} - (v^2 + 1 + v^{-2})E_nE_{n-1}E_n - (v^2 + 1 + v^{-2})E_{n-1}E_n^2 - E_{n-1}E_n^3 = 0, \\
(B3) & \quad E_iE_j = E_jE_i \text{ for } |i - j| > 1.
\end{align*}

Define the root vector $E_\alpha = \langle M_\alpha \rangle$ for $\alpha \in \Phi^+(B_n) = \Phi^+_1 \cup \Phi^+_2$. In particular, $E_\alpha = E_i$ for $1 \leq i \leq n$.

## The Hall algebra and the quantized Schur algebra: We are not going to define the quantized Schur algebra here. Instead we state the work of Green which is sufficient for our purpose. Let $\varphi$ be the algebra homomorphism from $H_v(A_n)$ to $S_v(n+1, r)$ defined in [4]. Then the kernel of $\varphi$ has a $\mathbb{Q}(v)$-basis

$$\left\{ \prod_{\alpha \in \Phi^+(A_n)} E^{(n\alpha)}_\alpha : \sum n_\alpha > r, \ n_\alpha \in \mathbb{N}_0 \right\},$$

where the product respects the ‘good’ order. We write $I_1(A_n)$ for the ideal $\text{Ker}(\varphi)$ of $H_v(A_n)$.

Now from the $\mathbb{Q}(v)$-algebra homomorphism $\varphi : H_v(A_n) \to S_v(n+1, r)$, we get naturally a $\mathbb{Q}(v)$-algebra homomorphism $\tilde{\varphi} : H_v(A_n) \to S_v(n+1, r)$, where $H_v(A_n)$ and $S_v(n+1, r)$ are obtained from $H_v(A_n)$ and $S_v(n+1, r)$ respectively with $v$ replaced by $v^2$ everywhere. We write $I_2(A_n)$ for the ideal $\text{Ker}(\tilde{\varphi})$ of $H_v(A_n)$. It is clear that $I_2(A_n)$ has a $\mathbb{Q}(v)$-basis of the same form as $I_1(A_n)$ with $v$ replaced by $v^2$ everywhere.

## Main results: We start with two lemmas about the positive roots of $A_n$ and $B_n$.

**Lemma 2.1.** (Type $A_n$) For a positive non-simple root $\alpha \in \Phi^+(A_n)$, there exist (unnecessarily unique) $\gamma_1, \gamma_2 \in \Phi^+(A_n)$ and a short exact sequence $0 \to M_{\gamma_1} \to M_\alpha \to M_{\gamma_2} \to 0$.

**Proof.** Write $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_r$ with $a + b + c = n$ and $b \geq 2$. Take $b_1, b_2 \in \mathbb{N}$ such that $b_1 + b_2 = b$. Set $\gamma_1 = \alpha_1 + b_1 + b_2 + c$ and $\gamma_2 = \alpha_1 + b_1 + b_2 + c$. Then $\alpha = \gamma_1 + \gamma_2,$ $(\gamma_1, \gamma_2) = 0,$ $(\gamma_2, \gamma_1) = -1,$ and $\gamma_1 \prec \gamma_2$. Now the lemma follows from the Auslander–Reiten quiver of the linearly oriented $A_n$. $\blacksquare$
Lemma 2.2. (Type $B_n$)

(1) For a positive and non-simple root $\alpha \in \Phi^+_1$, there exist (unnecessarily unique) $\gamma_1, \gamma_2 \in \Phi^+_1$ and a short exact sequence $0 \to M_{\gamma_1} \to M_\alpha \to M_{\gamma_2} \to 0$.

(2) For a positive root $\beta \in \Phi^+_2$, there exist uniquely $\gamma_1, \gamma_2 \in \Phi^+_1$ such that there is an Auslander–Reiten sequence $0 \to M_{\gamma_1} \to M_\alpha \to M_{\gamma_2} \to 0$.

Proof. (1) Suppose $\alpha = 0^a1^b0^c$ with $a + b + c = n$ and $b \geq 2$. Then $\gamma_1$ and $\gamma_2$ of the same form as in Lemma 2.1 will play the role. We have $\langle \gamma_1, \gamma_2 \rangle = 0$, $\langle \gamma_2, \gamma_1 \rangle = -2$.

(2) Suppose $\beta = 0^a1^b2^c$ with $a + b + c = n$ and $b, c > 0$. Take $\gamma_1 = 0^{a+b}1^c$ and $\gamma_2 = 0^a1^{b+c}$ in $\Phi^+_1$. Then $\beta = \gamma_1 + \gamma_2$, $\gamma_1 \prec \gamma_2$, $\langle \gamma_1, \gamma_2 \rangle = 1$ and $\langle \gamma_2, \gamma_1 \rangle = -1$. It is clear that such a pair $(\gamma_1, \gamma_2)$ is unique. The existence of the Auslander–Reiten sequence follows from the Auslander–Reiten quiver of $B_n$. $\square$

Our main results are the following three theorems:

Theorem 2.3. There exists a $\mathbb{Q}(v)$-algebra epimorphism $\pi : H_v(B_n) \to H_v(A_n)$ sending $E_i$ to $E_i$. The kernel of $\pi$ is an ideal of $H_v(B_n)$ generated by

$$E_n^2 E_{n-1} - (v^2 + v^{-2})E_n E_{n-1} E_n + E_{n-1} E_n^2,$$

and also generated by

$$\left\{ E_\beta - \frac{v^2 - 1}{v + v^{-1}} E_{\gamma_1} E_{\gamma_2} : \beta \in \Phi^+_2 \right\},$$

where $\gamma_1, \gamma_2 \in \Phi^+_1$ are determined uniquely by $\beta$ as in Lemma 2.2(2).

Fix any positive integer $r \in \mathbb{N}$.

Theorem 2.4. Write $I_2(B_n)$ for the ideal of the Hall algebra $H_v(B_n)$ generated by

$$\left\{ \prod_{\alpha \in \Phi^+_1, \beta \in \Phi^+_2} E_\alpha^{(n_\alpha)} E_\beta^{(n_\beta)} : \sum n_\alpha + 2 \sum n_\beta > r, n_\alpha, n_\beta \in \mathbb{N}_0 \right\},$$

where the product respects the ‘good’ order. Then the set of generators is actually a $\mathbb{Q}(v)$-basis of $I_2(B_n)$.

Consider the composition map $\psi : H_v(B_n) \to S_{v^2}(n + 1, r)$ of $\pi : H_v(B_n) \to H_v(A_n)$ and $\tilde{\psi} : H_v(A_n) \to S_{v^2}(n + 1, r)$.

Theorem 2.5. Write $I_1(B_n)$ for the $\mathbb{Q}(v)$-subspace of $H_v(B_n)$ with basis (the product respects the ‘good’ order)

$$\left\{ \prod_{\alpha \in \Phi^+_1} E_\alpha^{(n_\alpha)} : \sum n_\alpha > r, n_\alpha \in \mathbb{N}_0 \right\}.$$

The kernel of the composition map $\psi$ is the sum $I_2(B_n) + \text{Ker}(\pi)$ as an ideal of $H_v(B_n)$, and is the direct sum $I_1(B_n) \oplus \text{Ker}(\pi)$ as a $\mathbb{Q}(v)$-subspace of $H_v(B_n)$. 

3 Generalized Quantum Serre Relations

In this section, we prove some equalities in the Hall algebra \( H_v(A_n) \). Recall that the root vector \( E_\alpha = (M_\alpha) = \langle \alpha \rangle \) for a positive root \( \alpha \in \Phi^+(A_n) \), where \( \epsilon(\alpha) = \dim(\text{End}(M_\alpha)) - \dim(M_\alpha) = \langle \alpha, \alpha \rangle - \dim(M_\alpha) \). The Euler form \( \langle -, - \rangle \) and the symmetric Euler form \( (\cdot, \cdot) \) on the root lattice are defined by

\[
\langle \alpha, \beta \rangle = \langle \dim(M_\alpha), \dim(M_\beta) \rangle = \dim(\text{Hom}(M_\alpha, M_\beta)) - \dim(\text{Ext}(M_\alpha, M_\beta)), \\
(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle
\]

for any positive roots \( \alpha, \beta \in \Phi^+(A_n) \) (see [2]).

Lemma 3.1. For \( \alpha, \beta \in \Phi^+(A_n) \), the following are equivalent:

1. The symmetric Euler form \( (\alpha, \beta) = -1 \).
2. There exists a short exact sequence of the form \( 0 \to M_\alpha \to M_{\alpha+\beta} \to M_\beta \to 0 \)
   or \( 0 \to M_\beta \to M_{\alpha+\beta} \to M_\alpha \to 0 \).
3. The sum \( \alpha + \beta \) is again a positive root in \( \Phi^+(A_n) \).

Proof. (1)\(\Rightarrow\)(2) Note that for any \( \gamma_1, \gamma_2 \in \Phi^+(A_n) \), the Euler form \( \langle \gamma_1, \gamma_2 \rangle \in \{1, 0, -1\} \). Hence, \( (\alpha, \beta) = -1 \) if and only if either \( \langle \alpha, \beta \rangle = 0 \) and \( \langle \beta, \alpha \rangle = -1 \), or \( \langle \alpha, \beta \rangle = -1 \) and \( \langle \beta, \alpha \rangle = 0 \).

In the first case, from the properties of the ‘good’ order on \( \Phi^+(A_n) \), we have \( \text{Ext}(M_\alpha, M_\beta) = 0 \), \( \text{Hom}(M_\alpha, M_\beta) = 0 \), \( \text{Hom}(M_\beta, M_\alpha) = 0 \) and \( \dim(\text{Ext}(M_\beta, M_\alpha)) = 1 \). Then we obtain a non-split short exact sequence of the form \( 0 \to M_\alpha \to X \to M_\beta \to 0 \). Assume the middle term \( X \) is decomposable. Then there exists a nonzero proper direct summand \( X_1 \) such that the composition map \( M_\alpha \to X_1 \to M_\beta \) is nonzero. This is a contradiction with \( \text{Hom}(M_\alpha, M_\beta) = 0 \). Hence, \( X \) is indecomposable, and in particular, the dimension vector \( \dim(X) = \alpha + \beta \).

Similarly, the second case gives rise to a short exact sequence of the form \( 0 \to M_\beta \to M_{\alpha+\beta} \to M_\alpha \to 0 \).

(2)\(\Rightarrow\)(3) It is clear.

(3)\(\Rightarrow\)(1) Since \( \alpha + \beta \) is a positive root in \( \Phi^+(A_n) \), we have \( 1 = (\alpha + \beta, \alpha + \beta) = (\alpha, \alpha) + (\alpha, \beta) + (\beta, \beta) \). Also, \( 1 = (\alpha, \alpha) = (\beta, \beta) \). Hence, \( (\alpha, \beta) = -1 \).

Proposition 3.2. (Generalized quantum Serre relations) In \( H_v(A_n) \), we have the following generalized quantum Serre relations:

\[
E_\alpha^2E_\beta - (v + v^{-1})E_\alpha E_\beta E_\alpha + E_\beta E_\alpha^2 = 0
\]

for all positive roots \( \alpha, \beta \) satisfying \( (\alpha, \beta) = -1 \).

Proof. By definition, it suffices to prove for the star product that

\[
\]

Since \( (\alpha, \beta) = -1 \), we have either \( \langle \alpha, \beta \rangle = 0 \) and \( \langle \beta, \alpha \rangle = -1 \), or \( \langle \alpha, \beta \rangle = -1 \) and \( \langle \beta, \alpha \rangle = 0 \).
In the first case, by Lemma 3.1, we have a short exact sequence $0 \to M_\alpha \to M_{\alpha+\beta} \to M_\beta \to 0$. Hence,

$$[M_\alpha]^2 * [M_\beta] = (v(v^2 + 1)[M_\alpha \oplus M_\beta]) \circ [M_\beta]$$
$$= v(v^2 + 1)[M_\alpha \oplus M_\alpha \oplus M_\beta],$$

$$[M_\alpha] * [M_\beta] * [M_\alpha] = ([M_\alpha] \circ [M_\beta]) * [M_\alpha] = [M_\alpha \oplus M_\beta] * [M_\alpha]$$
$$= [M_\alpha \oplus M_\beta] \circ [M_\alpha]$$
$$= (v + 1)[M_\alpha \oplus M_\beta \oplus M_\alpha] + [M_\alpha \oplus M_{\beta+\alpha}],$$

$$[M_\beta] * [M_\alpha]^2 = v(v^2 + 1)[M_\beta] \circ [M_\alpha]$$
$$= v(v^2 + 1)v^{-2}[M_\beta] \circ [M_\alpha]$$
$$= (v + v^{-1})([M_\beta \oplus M_\alpha \oplus M_\alpha] + [M_{\alpha+\beta} \oplus M_\alpha]).$$

Then the relation (S) follows.

Now assume $\langle \alpha, \beta \rangle = -1$ and $\langle \beta, \alpha \rangle = 0$. By Lemma 3.1, we have a short exact sequence $0 \to M_\beta \to M_{\alpha+\beta} \to M_\alpha \to 0$. Hence,

$$[M_\alpha]^2 * [M_\beta] = (v(v^2 + 1)v^{-2}[M_\alpha \oplus M_\alpha]) \circ [M_\beta]$$
$$= (v + v^{-1})([M_\alpha \oplus M_\alpha \oplus M_\beta] + [M_\alpha \oplus M_{\alpha+\beta}]),$$

$$[M_\alpha] * [M_\beta] * [M_\alpha] = v^{-1}([M_\alpha] \circ [M_\beta]) * [M_\alpha]$$
$$= v^{-1}([M_\alpha \oplus M_\beta] + [M_{\alpha+\beta}]) * [M_\alpha]$$
$$= v^{-1}v([M_\alpha \oplus M_\beta] + [M_{\alpha+\beta}]) \circ [M_\alpha]$$
$$= (v^2 + 1)([M_\alpha \oplus M_\beta \oplus M_\alpha] + [M_{\alpha+\beta} \oplus M_\alpha]),$$

$$[M_\beta] * [M_\alpha]^2 = v(v^2 + 1)[M_\beta] \circ [M_\alpha]$$
$$= v(v^2 + 1)[M_\beta] \circ [M_\alpha]$$
$$= v(v^2 + 1)[M_\beta \oplus M_\alpha \oplus M_\alpha].$$

The relation (S) follows similarly.

Note that the quantum Serre relation (A1) in Section 2 is a special case of the generalized quantum Serre relation (S).

**Proposition 3.3.** Let $\alpha, \beta, \gamma \in \Phi^+(A_n)$ be three positive roots with $\alpha = \beta + \gamma$ and $\beta < \gamma$. Then it holds in $H_v(A_n)$ that $E_\alpha = E_\beta E_\gamma - v^{-1}E_\beta E_\gamma$ and $E_\alpha E_\gamma = v E_\beta E_\alpha$.

**Proof.** By Lemma 3.1, there exists a short exact sequence $0 \to M_\beta \to M_\alpha \to M_\gamma \to 0$ and $\langle \beta, \gamma \rangle = 0$, $\langle \gamma, \beta \rangle = -1$. By the definition of the product in the Hall algebra $H_v(A_n)$, we have

$$[M_\beta] * [M_\gamma] = [M_\beta] \circ [M_\gamma] = [M_\beta \oplus M_\gamma],$$

$$[M_\alpha] * [M_\beta] = v^{-1}[M_\alpha] \circ [M_\beta] = v^{-1}([M_\beta \oplus M_\alpha] + [M_\gamma]).$$

Hence,

Notice that $\dim(M_\alpha) = \dim(M_\beta) + \dim(M_\gamma)$ and $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = \langle \gamma, \gamma \rangle = 1.$ Since the root vector $E_\alpha = \langle M_\alpha \rangle = \nu^{(\alpha)}[M_\alpha]$ and $\epsilon(\alpha) = \langle \alpha, \alpha \rangle - \dim(M_\alpha),$ it holds that $E_\alpha = E_\gamma E_\beta - v^{-1} E_\beta E_\gamma.$

Now from Proposition 3.2, we have the generalized quantum Serre relation

$$E_\alpha E_\beta - (v + v^{-1}) E_\beta E_\alpha + E_\beta^2 E_\gamma = 0.$$  

Hence,

$$E_\alpha E_\beta = (E_\gamma E_\beta - v^{-1} E_\beta E_\gamma) E_\beta = E_\gamma E_\beta^2 - v^{-1} E_\beta E_\gamma E_\beta$$

$$= v E_\beta E_\gamma E_\beta - E_\beta^2 E_\gamma = v E_\beta (E_\gamma E_\beta - v^{-1} E_\beta E_\gamma) = v E_\beta E_\alpha,$$

which completes the proof.

\section*{4 Proof of Main Results}

The root systems $\Phi^+(A_n)$ and $\Phi^+(B_n) = \Phi^+_1 \cup \Phi^+_2$ are described in Section 2. Recall that $\Phi^+_1 = \Phi^+(A_n)$ by identifying the simple roots.

\begin{lemma}
Let $\alpha = 0^a 1^b 0^c,$ $\gamma_1 = 0^a 1^b 1^c 0^d,$ $\gamma_2 = 0^1 1^b 0^2 0^c \in \Phi^+_1,$ where $a, c \in \mathbb{N}_0$ and $b, b_1, b_2 \in \mathbb{N}$ such that $a + b + c = n$ and $b_1 + b_2 = b.$ In the Hall algebra $H_v(A_n),$ we have $E_\alpha = E_\gamma E_\beta - v^{-1} E_\beta E_\gamma,$ and in the Hall algebra $H_v(B_n),$ we have $E_\alpha = E_\gamma E_\beta - v^{-2} E_{\gamma_1} E_{\gamma_2}.$

\begin{proof}
Clearly, $\alpha = \gamma_1 + \gamma_2,$ and $\gamma_1 \prec \gamma_2$ with respect to the ‘good’ order. The equality in $H_v(A_n)$ now follows from Proposition 3.3.

Consider the Euler form on $\Phi^+(B_n).$ We have $\langle \gamma_1, \gamma_2 \rangle = 0,$ $\langle \gamma_2, \gamma_1 \rangle = -2,$ $\langle \gamma_2, \gamma_2 \rangle = 2,$ and

$$\langle \gamma_1, \gamma_1 \rangle = \langle \alpha, \alpha \rangle = \begin{cases} 2 & \text{when } c = 0, \\ 1 & \text{when } c \neq 0. \end{cases}$$

Also, $\dim(M_\alpha) = \dim(M_\gamma) + \dim(M_\beta).$ Hence, $\epsilon(\alpha) + 2 = \epsilon(\gamma_1) + \epsilon(\gamma_2).$

By Lemma 2.2(1), there is a short exact sequence $0 \rightarrow M_{\gamma_1} \rightarrow M_\alpha \rightarrow M_{\gamma_2} \rightarrow 0.$ In the Hall algebra $H_v(A_n),$ we have

$$[M_{\gamma_1}] * [M_{\gamma_2}] = [M_{\gamma_1}] \circ [M_{\gamma_2}] = [M_{\gamma_1} \circ M_{\gamma_2}].$$

Hence, $[M_{\gamma_1}] = v^2 [M_{\gamma_2}] * [M_{\beta}] - [M_{\beta}] * [M_{\gamma_2}].$ The equality $E_\alpha = E_{\gamma_2} E_{\gamma_1} - v^{-2} E_{\gamma_1} E_{\gamma_2}$ follows from the definition that $E_\alpha = v^{\epsilon(\alpha)}[M_\alpha].$
\end{proof}

\begin{lemma}
Let $\beta = 0^a 1^b 2^c \in \Phi^+_2,$ and $\gamma_1 = 0^a 1^b 1^c,$ $\gamma_2 = 0^a 1^{b+c} \in \Phi^+_1.$ In the Hall algebra $H_v(A_n),$ we have $E_{\gamma_2} E_{\gamma_1} = v E_{\gamma_1} E_{\gamma_2},$ and in the Hall algebra $H_v(B_n),$ we have $E_\beta = \frac{1}{v+v^{-1}} (E_{\gamma_2} E_{\gamma_1} - E_{\gamma_1} E_{\gamma_2}).$

\begin{proof}
Let $\gamma_3 = 0^a 1^b 0^c \in \Phi^+(A_n).$ Then $\gamma_2 = \gamma_1 + \gamma_3$ and $\gamma_1 \prec \gamma_3.$ The relation in $H_v(A_n)$ follows from Proposition 3.3.

For the relation in $H_v(B_n),$ note that $\beta = \gamma_1 + \gamma_2$ and $\dim(\text{End}(M_\beta)) = 2,$ $\dim(\text{End}(M_\gamma)) = \dim(\text{End}(M_{\gamma_1})) = 1,$ $\dim(M_\beta) = \dim(M_{\gamma_1}) + \dim(M_{\gamma_2}).$ Thus,
\( \epsilon_{\beta} = \epsilon_{\gamma_1} + \epsilon_{\gamma_2}. \) From \( \mathcal{E}_\alpha = \langle M_\alpha \rangle = v^{\epsilon_{\alpha}} [M_\alpha] \) for any positive root \( \alpha, \) it suffices to prove the relation

\[
(R) \quad [M_\beta] = \frac{1}{v + v^{-1}} ([M_{\gamma_2}] * [M_{\gamma_1}] - [M_{\gamma_1}] * [M_{\gamma_2}]).
\]

By Lemma 2.2(2), we have the Auslander–Reiten sequence \( 0 \rightarrow M_{\gamma_1} \rightarrow M_\beta \rightarrow M_{\gamma_2} \rightarrow 0, \) and \( \langle \gamma_1, \gamma_2 \rangle = 1, \langle \gamma_2, \gamma_1 \rangle = -1. \) Hence, we have \( \text{Hom}(M_{\gamma_2}, M_{\gamma_1}) = 0, \text{Ext}(M_{\gamma_1}, M_{\gamma_2}) = 0, \dim(\text{Ext}(M_{\gamma_1}, M_{\gamma_2})) = 1, \dim(\text{Hom}(M_{\gamma_1}, M_{\gamma_2})) = 1 \) and \( M_{\gamma_1} \) is a submodule of \( M_{\gamma_2}. \) Then

\[
[M_{\gamma_2}] * [M_{\gamma_1}] = v^{-1} [M_{\gamma_2}] \circ [M_{\gamma_1}] = v^{-1} (v^2 [M_{\gamma_2} \oplus M_{\gamma_1}] + (v^2 + 1) [M_\beta]),
\]

\[
[M_{\gamma_1}] * [M_{\gamma_2}] = v^1 [M_{\gamma_1}] \circ [M_{\gamma_2}] = v [M_{\gamma_1} \oplus M_{\gamma_2}].
\]

The relation \( (R) \) follows now. \( \square \)

From the description of the Hall algebras \( H_{v^2}(A_n) \) and \( H_v(B_n) \) by quantum Serre relations in Section 2, one sees directly that sending \( \mathcal{E}_1 \) to \( E_i \) \( (i = 1, 2, \ldots, n) \) provides an algebra epimorphism \( \pi : H_v(B_n) \rightarrow H_{v^2}(A_n). \)

**Lemma 4.3.** The images of the root vectors \( \mathcal{E}_\alpha \) \( (\alpha \in \Phi^+_1) \) and \( \mathcal{E}_\beta \) \( (\beta \in \Phi^+_2) \) of \( H_v(B_n) \) under \( \pi \) are \( \pi(\mathcal{E}_\alpha) = E_\alpha \) and \( \pi(\mathcal{E}_\beta) = \frac{v^2 - 1}{v + v^{-1}} E_{\gamma_1} E_{\gamma_2}, \) where \( \gamma_1, \gamma_2 \in \Phi^+_1 \) are uniquely determined by \( \beta \in \Phi^+_2 \) as in Lemma 2.2(2).

**Proof.** Suppose \( \alpha \in \Phi^+_1. \) By Lemma 4.1 and induction on \( \alpha, \) it is clear that the expression of \( \mathcal{E}_\alpha \in H_v(B_n) \) into \( \mathcal{E}_i \) is the same as the expression of \( E_\alpha \in H_{v^2}(A_n) \) into \( E_i. \) Then from \( \pi(\mathcal{E}_i) = E_i, \) it follows that \( \pi(\mathcal{E}_\alpha) = E_\alpha. \)

Suppose \( \beta \in \Phi^+_2. \) By Lemma 4.2 and the relation we obtained just now, we have \( \pi(\mathcal{E}_\beta) = \pi \left( \frac{1}{v + v^{-1}} (E_{\gamma_2} E_{\gamma_1} - E_{\gamma_1} E_{\gamma_2}) \right) = \frac{1}{v + v^{-1}} (E_{\gamma_2} E_{\gamma_1} - E_{\gamma_1} E_{\gamma_2}) = \frac{v^2 - 1}{v + v^{-1}} E_{\gamma_1} E_{\gamma_2}, \) which completes the proof. \( \square \)

We are now ready to prove our main results.

**Proof of Theorem 2.3.** Note that

\[
\begin{align*}
E_n^2 E_{n-1} &- (v^2 + 1 + v^{-2}) E_n^2 E_{n-1} E_n + (v^2 + 1 + v^{-2}) E_n E_{n-1} E_n^2 - E_{n-1} E_n^3 \\
&= E_n \left( E_n^2 E_{n-1} - (v^2 + v^{-2}) E_n E_{n-1} E_n + E_{n-1} E_n^2 \right) \\
&\quad - (E_n^2 E_{n-1} - (v^2 + v^{-2}) E_n E_{n-1} E_n + E_{n-1} E_n^2) E_n.
\end{align*}
\]

So the kernel of \( \pi \) is generated by \( \mathcal{E}_n^2 E_{n-1} - (v^2 + v^{-2}) \mathcal{E}_n \mathcal{E}_{n-1} \mathcal{E}_n + \mathcal{E}_{n-1} \mathcal{E}_n^2. \) We write \( I_3(B_n) \) for the ideal of \( H_v(B_n) \) generated by

\[
\left\{ \mathcal{E}_\beta - \frac{v^2 - 1}{v + v^{-1}} E_{\gamma_1} E_{\gamma_2} : \beta \in \Phi^+_2 \right\},
\]

where \( \gamma_1, \gamma_2 \in \Phi^+_1 \) are determined by \( \beta \) as in Lemma 2.2(2). By Lemma 4.3, the ideal \( I_3(B_n) \) is contained in the kernel of \( \pi. \)
Consider $\beta = 0^{n-2}121 = \alpha_{n-1} + 2\alpha_n \in \Phi_2^+$. It uniquely determines $\gamma_1 = \alpha_n$, $\gamma_2 = \alpha_{n-1} + \alpha_n \in \Phi_1^+$. In $H_v(B_n)$, we have $E_\beta = \frac{1}{v^2 + v^{-1}}(E_{\gamma_1}E_{\gamma_1} - E_{\gamma_2}E_{\gamma_2})$ by Lemma 4.2, and $E_{\gamma_2} = E_{\gamma_1}E_n - v^{-2}E_nE_{\gamma_2}$ by Lemma 4.1. Therefore, in $I_3(B_n)$,

$$E_{\beta} = \frac{v^2 - 1}{v + v^{-1}}E_{\gamma_1}E_{\gamma_2} = \frac{1}{v + v^{-1}}(E_{\gamma_1}E_{\gamma_1} - E_{\gamma_2}E_{\gamma_2}) - \frac{v^2 - 1}{v + v^{-1}}E_{\gamma_1}E_{\gamma_2} = \frac{E_{\gamma_1}E_{\gamma_2} - v^2E_{\gamma_1}E_{\gamma_2}}{v + v^{-1}} = \frac{E_{\gamma_1}E_{\gamma_2} - (v^2 + v^{-2})E_nE_{\gamma_2}}{v + v^{-1}} = \frac{E_{\gamma_2} - (v^2 + v^{-2})E_nE_{\gamma_2}}{v + v^{-1}}.$$ 

It follows that $E_{n-1}E_n^2 - (v^2 + v^{-2})E_nE_{n-1} + E_n^2E_{n-1}$, and hence Ker($\pi$) lies in $I_3(B_n)$. Therefore, Ker($\pi$) = $I_3(B_n)$. □

**Proof of Theorem 2.4.** Define the degree function on the root vectors and on the PBW-basis of $H_v(B_n)$ as follows:

$$\deg(E_{\alpha}) = \begin{cases} 1 & \text{if } \alpha \in \Phi_1^+, \\ 2 & \text{if } \alpha \in \Phi_2^+, \end{cases}$$

and for $\gamma_1, \gamma_2 \in \Phi^+(B_n)$ with $\gamma_1 \preceq \gamma_2$,

$$\deg(E_{\gamma_1}E_{\gamma_2}) = \deg(E_{\gamma_1}) + \deg(E_{\gamma_2}).$$

For any element $E \in H_v(B_n)$, define deg($E$) to be the minimal degree of the PBW-basis elements which have non-zero coefficients in $E$.

For a positive integer $r$, let $V_r$ be the subspace of $H_v(B_n)$ with a $\mathbb{Q}(v)$-basis

$$\left\{ \prod_{\alpha \in \Phi_1^+, \beta \in \Phi_2^+} E_{\alpha}^{n_{\alpha}}E_{\beta}^{n_{\beta}} : \sum n_{\alpha} + 2\sum n_{\beta} > r \right\}.$$ 

To see that $V_r$ is an ideal, it suffices to show that for any positive root $\alpha \in \Phi^+(B_n)$ and any word $E_\gamma E_{\gamma_2} \cdots E_{\gamma_m}$ in the PBW-basis of $H_v(B_n)$ ($\gamma_1 \preceq \gamma_2 \preceq \cdots \preceq \gamma_m$), we have

$$\deg(E_\alpha E_\gamma E_{\gamma_2} \cdots E_{\gamma_m}) \geq \deg(E_\gamma E_{\gamma_2} \cdots E_{\gamma_m}),$$

$$\deg(E_\gamma E_{\gamma_2} \cdots E_{\gamma_m}) \leq \deg(E_\gamma E_{\gamma_2} \cdots E_{\gamma_2} E_{\gamma_m}).$$

We shall prove the first inequality only.

If $\alpha \preceq \gamma_1$, it is well ordered already and

$$\deg(E_\alpha E_\gamma E_{\gamma_2} \cdots E_{\gamma_m}) = \deg(E_\alpha) + \deg(E_\gamma E_{\gamma_2} \cdots E_{\gamma_m}) \leq \deg(E_\gamma E_{\gamma_2} \cdots E_{\gamma_m}).$$

If $\gamma_1 \preceq \alpha$ and $\langle \alpha, \gamma_1 \rangle = 0$, then $\Ext(M_\alpha, M_{\gamma_1}) = 0$ and $E_\alpha E_{\gamma_1} = v^{-\langle \gamma_1, \alpha \rangle}E_{\gamma_1}E_\alpha$. We see that the degree is preserved in this order-changing.

If $\gamma_1 \preceq \alpha$ and $\langle \alpha, \gamma_1 \rangle \neq 0$, then $\Ext(M_{\gamma_2}, M_{\gamma_1}) \neq 0$ and there exists a short exact sequence of the form $0 \to M_{\gamma_1} \to X \to M_\alpha \to 0$, so the dimension vector of $X$ is $\gamma_1 + \alpha$. 
If $X$ is decomposable, say $X = \bigoplus M_{\beta_i}$ with $\beta_i \in \Phi^+(B_n)$ satisfying $\sum_i \beta_i = \gamma_1 + \alpha$ and $\gamma_1 < \beta_i < \alpha$. Then we have

$$\mathcal{E}_\alpha \mathcal{E}_{\gamma_1} = c_1 \mathcal{E}_{\gamma_1} \mathcal{E}_\alpha + c_2 \prod_{i} \mathcal{E}_{\beta_i}$$

with coefficients $c_1, c_2 \in \mathbb{Q}(v)$. From the Auslander–Reiten quiver of $B_n$, one sees that $\deg(\prod_{i} \mathcal{E}_{\beta_i}) = \deg(\mathcal{E}_\alpha \mathcal{E}_{\gamma_1})$, and that for any $\beta_i$, there is no $\gamma \in \Phi^+(B_n)$ with $\gamma_1 \preceq \gamma$ and $\text{Ext}(M_{\beta_i}, M_{\gamma}) \neq 0$.

If $X$ is indecomposable, suppose $X = M_\beta$ with $\beta = \gamma_1 + \alpha$. We have

$$\mathcal{E}_\alpha \mathcal{E}_{\gamma_1} = d_1 \mathcal{E}_{\gamma_1} \mathcal{E}_\alpha + d_2 \mathcal{E}_\beta$$

with coefficients $d_1, d_2 \in \mathbb{Q}(v)$. Since $M_{\gamma_1}$ is a submodule of $M_\beta$, we have $\deg(\mathcal{E}_\beta) \geq \mathcal{E}_{\gamma_1}$. There is at most one positive root $\gamma \in \Phi^+(B_n)$ with $\gamma_1 \preceq \gamma$ such that $\text{Ext}(M_\beta, M_\gamma) \neq 0$, which happens only when $\gamma = \gamma_1$ and $\beta = \alpha^a 1^b \in \Phi_1^+$ with $a > 0$ and $a + b = n$. In this case, we have short exact sequences

$$0 \to M_{\gamma_1} \to M_\beta \to M_\alpha \to 0,$$

$$0 \to M_{\gamma_1} \to M_{\beta + \gamma_1} \to M_\beta \to 0.$$

Moreover,

$$\deg(\mathcal{E}_\beta) = \deg(\mathcal{E}_{\gamma_1}) = 1,$$

$$\deg(\mathcal{E}_{\beta + \gamma_1}) = 2 = \deg(\mathcal{E}_{\gamma_1} \mathcal{E}_\alpha) = \deg(\mathcal{E}_{\gamma_1}^2).$$

With these facts, by induction on the length $m$ of the word $\mathcal{E}_{\gamma_1} \mathcal{E}_{\gamma_2} \cdots \mathcal{E}_{\gamma_m}$, we complete the proof. \hfill \Box

Now Theorem 2.5 is just a direct consequence.

**Proof of Theorem 2.5.** Recall that the algebra map $\psi : H_v(B_n) \to S_{vz}(n + 1, r)$ is the composition of $\pi : H_v(B_n) \to H_v(A_n)$ and $\tilde{\varphi} : H_v(A_n) \to S_{vz}(n + 1, r)$. We know from Section 2 that Ker($\tilde{\varphi}$) = $I_2(A_n)$. By Lemma 4.3, the map $\pi$ sends the ideal $I_2(B_n)$ of $H_v(B_n)$ to the ideal $I_2(A_n)$ of $H_v(A_n)$. Therefore,

$$\text{Ker}(\psi) = \text{Ker}(\tilde{\varphi} \circ \pi) = \pi^{-1}(\text{Ker}(\tilde{\varphi})) = \pi^{-1}(I_2(A_n)) = I_2(B_n) + \text{Ker}(\pi).$$

Note that although $I_1(B_n)$ is not an ideal of the Hall algebra $H_v(B_n)$, the image of $I_1(B_n)$ under $\pi$ is exactly $I_2(A_n)$, and they have the same dimension over $\mathbb{Q}(v)$. Hence, Ker($\psi$) decomposes into the direct sum of $I_1(B_n)$ and Ker($\pi$) as a $\mathbb{Q}(v)$-vector space. \hfill \Box

**References**


