Lattices of Infinite Rank and Their Decompositions

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1. A first example

Let $L = \mathbb{Z}^{(I)}$ be any free abelian group with an involution, i. e. a representation $\mathbb{Z}C_2 \to \mathrm{End}(L)$. Problem: Classify such objects!

Answer: $L = L^+ \oplus L^-$.

More generally, for p prime, Butler, Campbell, and Kovács showed (2004) that every "large" $\mathbb{Z}C_p$-lattice L decomposes into $\mathbb{Z}C_p$-lattices: If $\mathcal{O} = \mathbb{Z}[\sqrt[p]{1}]$ has class number h, there are $2h+1$ indecomposables (Diederichsen-Reiner 1957), namely

$$\begin{cases} \mathbb{Z}; \quad h \text{ ideal classes } I \text{ of } \mathcal{O}; \\ h \text{ extensions } I \rightarrowtail E \twoheadrightarrow \mathbb{Z} \ (\curvearrowright E \text{ projective}). \end{cases}$$

2. Large lattices over orders

Let R be a Dedekind domain with quotient field K, Λ an R-*order*, an R algebra with ${}_R\Lambda$ f. g. projective.

Λ-**Lat** := Λ-**Mod** $\cap$ R-**Proj** (*large* Λ-*lattices*)
Λ-**lat** := Λ-**Mod** $\cap$ R-**proj** (ordinary Λ-lattices)

Problem: Which R-orders Λ have the property

(FD) Every $L \in \Lambda$-**Lat** decomposes into Λ-lattices

Such L are called *fully decomposable (f. d.)*.

Localization: For $\mathfrak{p} \in \operatorname{Spec} R$, every $\Lambda_{\mathfrak{p}}$-lattice E has a (unique) decomposition $E = \bigoplus E_i$ into indecomposable $\Lambda_{\mathfrak{p}}$-lattices E_i (Krull-Schmidt).

It is enough to consider the set $\mathrm{S}(\Lambda)$ of *singular* primes $\mathfrak{p} \neq 0$ (i. e. where $\Lambda_{\mathfrak{p}}$ is not hereditary).

$E, F \in \Lambda$-**lat** are in the same *genus* if $E_{\mathfrak{p}} \cong F_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{S}(\Lambda)$. (Then $E \in \operatorname{ind} \Lambda_{\mathfrak{p}} \Leftrightarrow F \in \operatorname{ind} \Lambda_{\mathfrak{p}}$.)

For $\Lambda = \mathbb{Z}C_p$, we have $\mathrm{S}(\Lambda) = \{(p)\}$, and there are three indecomposable genera:

$$\operatorname{ind} \Lambda_{(p)} = \{\mathbb{Z}_p, \mathcal{O}_p, \Lambda_p\}.$$
$$1 \;+\; h \;+\; h$$

3. Local case $\mathfrak{p} = 0$

Here $\Lambda_{\mathfrak{p}} = K\Lambda =: A$ finite dimensional K-algebra.

Ringel, Tachikawa 1975: $\quad |\operatorname{ind} A| < \infty \;\Rightarrow\;$ (FD)
Auslander 1976: $\quad \Leftarrow$

Proof. Assume (FD). Let $M \in \operatorname{ind} A$ and e split monomorphism.

$$\begin{array}{ccc} M & \overset{e}{\underset{\text{direct}}{\rightarrowtail}} & \prod \operatorname{ind} A \\ \downarrow & & \downarrow p_i \\ \vartheta^{-} M & \dashrightarrow & M_i \end{array}$$

$\curvearrowright \exists i \colon p_i e$ invertible. $\curvearrowright \prod \operatorname{ind} A = \coprod \operatorname{ind} A$. $\curvearrowright |\operatorname{ind} A| < \infty$. $\square$

Note: If $|\operatorname{ind} A| = \infty$, there even exist large indecomposables.

4. The case: $A = K\Lambda$ non-semisimple

Theorem 1. *Let $A/\operatorname{Rad} A$ be separable, and let $\operatorname{Rad} A_{\mathfrak{p}} \neq 0$ for some $\mathfrak{p} \neq 0$.* $\curvearrowright$ $\exists L \in \Lambda$-**Lat** *non-f. d., and a f. d. $L' \subset L$ with* $\dim K(L/L') < \infty$.

Example 2. $\Lambda = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$, with representations $\begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$, where $L_1 \to L_2$ in $\mathbb{Z}$-**Lat**. Then

$$\operatorname{ind} \Lambda = \left\{ \begin{pmatrix} \mathbb{Z} \\ 0 \end{pmatrix}; \begin{pmatrix} n\mathbb{Z} \\ \mathbb{Z} \end{pmatrix}, n \in \mathbb{N} \right\}.$$

There are no large indecomposables!
But (FD) does not hold: Choose $L_2 \twoheadrightarrow \mathbb{Q}$ (p prime):

$$\begin{array}{ccccc}
L_3 & \hookrightarrow & L_1 & \twoheadrightarrow & \mathbb{Z}_p \\
\| & & \downarrow & \text{ex.} & \downarrow \\
L_3 & \hookrightarrow & L_2 & \twoheadrightarrow & \mathbb{Q} \\
 & & \downarrow & & \downarrow \\
 & & \mathbb{Z}_{p^\infty} & = & \mathbb{Z}_{p^\infty}
\end{array}
\qquad
\underset{\text{fully dec.}}{\begin{pmatrix} L_3 \\ L_3 \end{pmatrix}} \hookrightarrow \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \twoheadrightarrow \underset{\text{rank 1}}{\begin{pmatrix} \mathbb{Z}_p \\ \mathbb{Q} \end{pmatrix}}$$

Theorem 2. *Let $A/\operatorname{Rad} A$ be separable, and let $\operatorname{Rad} A \neq 0$. Then* Λ-**Lat** *has a subfactor* Δ-**Mod**, *where Δ is a maximal order in a skew-field.*

Example 3. $\Lambda = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$. Then Λ-$\overline{\mathbf{Lat}} \approx \mathbf{Ab}$.
Indecomposables in Λ-**Lat** are finitely generated, while **Ab** has arbitrary large indecomposables, but not many finitely generated objects!

5. Local case $\mathfrak{p} \neq 0$ ($K\Lambda_{\mathfrak{p}}$ semisimple)

Assume: R is a complete discrete valuation domain. For $M \in A$-**Mod**, define

$$\mathfrak{L}_\Lambda(M) := \{L \in \Lambda\text{-}\mathbf{Lat} \mid KL = M\}.$$

If $\dim_K M = \infty$, $\exists L_1, L_2 \in \mathfrak{L}_\Lambda(M)\colon L_1 + L_2 = M$.

BCK 2004: $\quad |\mathrm{ind}\,\Lambda| < \infty \;\Rightarrow\; \text{(FD)}.$

Proof. Let $E := \bigoplus \mathrm{ind}\,\Lambda$, and $\Gamma := \mathrm{End}_\Lambda(E)^{\mathrm{op}}$ (Auslander order). Then Λ-**Lat** $\approx \Gamma$-**Proj**. Since Γ is semiperfect, this implies (FD). □

Converse: (Product argument does not work!)
$\mathsf{M}(\Lambda\text{-}\mathbf{lat}) := \{\text{2-termed complexes}\}/\text{homotopy}$

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_1 & \xrightarrow{a} & E_0 & \longrightarrow & 0 \\ \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\ 0 & \longrightarrow & F_1 & \xrightarrow[b]{} & F_0 & \longrightarrow & 0 \end{array} \qquad \begin{array}{l} a_+ := E_0 \\ a_- := E_1 \end{array}$$

Assume (FD). $\curvearrowright$ $K\Lambda$ semisimple. $\curvearrowright$ Λ-**lat** has almost split sequences. $\curvearrowright$ $\mathsf{M}(\Lambda\text{-}\mathbf{lat})$ has L-functors:

$$\mathsf{M}(\Lambda\text{-}\mathbf{lat}) \rightleftarrows \mathsf{M}(\Lambda\text{-}\mathbf{lat}) \qquad L \dashv L^-$$

L^- is pointed: $a \xrightarrow{\lambda_a^-} L^-a \qquad$ Dually: $La \xrightarrow{\lambda_a} a.$

Explicitly: $$\begin{array}{ccc} E_1' & \longrightarrow & E_1 \\ \Big\downarrow La & \text{ex.} & \Big\downarrow a \\ E_0' & \longrightarrow & E_0 \end{array}$$

Mapping cone $\asymp$ almost split sequence $E_1' \rightarrowtail E_0' \oplus E_1 \twoheadrightarrow E_0$.

Define $a \in \mathrm{Fix}\,L :\Leftrightarrow \lambda_a$ invertible $\Leftrightarrow E_0 \in \Lambda$-**proj**.

The (Grothendieck) localization is abelian:

$$\mathsf{M}(\Lambda\text{-}\mathbf{lat})/\operatorname{Fix} L \approx \operatorname{Ext}(\Lambda\text{-}\mathbf{lat}).$$

We call $\mathsf{M}(\Lambda\text{-}\mathbf{lat})$ *L-finite* if for each object a, the ladders $L^n a, L^{-n} a$ become stationary.

For $\operatorname{Rad}\Lambda \overset{u}{\hookrightarrow} \Lambda$ in $\mathsf{M}(\Lambda\text{-}\mathbf{lat})$, the right ladder

$$u \overset{\lambda^-}{\to} L^- u \overset{\lambda^-}{\to} L^{-2} u \overset{\lambda^-}{\to} L^{-3} u \overset{\lambda^-}{\to} \cdots$$

is *invertible*, i. e. $L^{-n}u = LL^{-(n+1)}u$, for all $n \in \mathbb{N}$. Dually, there is an invertible left ladder, starting with $\Lambda^* \overset{v}{\hookrightarrow} \operatorname{Rad}^\circ(\Lambda^*)$, where $\operatorname{Rad}^\circ \dashv \operatorname{Rad}$.

Theorem 3. *The following are equivalent.*

(a) $|\operatorname{ind}\Lambda| < \infty$.

(b) $\mathsf{M}(\Lambda\text{-}\mathbf{lat})$ *is L-finite.*

(c) *There is some $n \in \mathbb{N}$ with $L^{-n}u \cong v$.*

Proof. (a) $\Rightarrow$ (b): Harada-Sai, (b) $\Rightarrow$ (c): trivial. (c) $\Rightarrow$ (a): For $E \in \operatorname{ind}\Lambda$, choose $e\colon E \hookrightarrow E'$ in Λ-**lat** with E'/E simple. $\curvearrowright$ $\exists$ projective cover p:

$$\begin{array}{ccccc} \operatorname{Rad} P & \overset{u_P}{\hookrightarrow} & P & \overset{p}{\twoheadrightarrow} & E'/E \\ \downarrow & \text{ex.} & \downarrow & & \| \\ E & \overset{e}{\hookrightarrow} & E' & \twoheadrightarrow & E'/E \end{array}$$

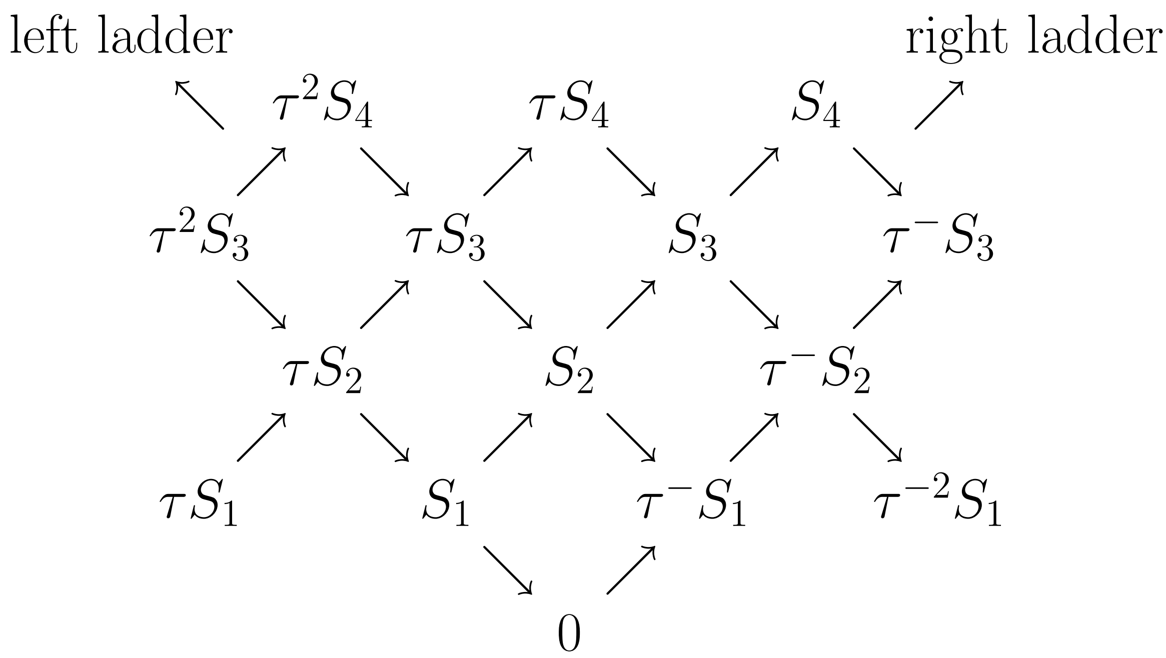
This gives a morphism $\alpha\colon u_P \to e$.

It follows that $E \in \operatorname{add} \bigoplus_{i=0}^{n} (L^{-i}u_P)_-$. $\square$

6. Large indecomposables

L-functors can be used for the construction of large indecomposables.

Example 4. Let A be a finite dimensional K-algebra. Assume that the Auslander-Reiten quiver has a quasi-serial component (e. g., a tube):



Then $\varprojlim \tau^{n-1} S_n$ and $\varinjlim S_n$ are indecomposable.

We return to an R-order Λ (local case).

Lemma. *Let $K\Lambda$ be semisimple, $E_i \in \operatorname{ind} \Lambda$, and*

$$\begin{array}{l} E_0 \xrightarrow{e_0} E_1 \xrightarrow{e_1} E_2 \xrightarrow{e_2} E_3 \xrightarrow{e_3} \cdots \\ \big\downarrow f \\ F \end{array} \qquad F \dashrightarrow E_3$$

a sequence of non-split epimorphisms. $\curvearrowright$ Every monomorphism f is a factor of some $e_n \cdots e_0$.

Theorem 4. *Let $K\Lambda$ be semisimple, $|\operatorname{ind}\Lambda| = \infty$. Then there exists a large indecomposable Λ-lattice.*

Proof. With $\operatorname{Rad}\Lambda \overset{u}{\hookrightarrow} \Lambda$, there is an infinite invertible ladder $u \xrightarrow{\lambda^-} L^- u \xrightarrow{\lambda^-} L^{-2}u \xrightarrow{\lambda^-} L^{-3}u \xrightarrow{\lambda^-} \cdots$. Using the Lemma, construct a sequence of kernels

$$E_0 \overset{e_0}{\rightarrowtail} E_1 \overset{e_1}{\rightarrowtail} E_2 \overset{e_2}{\rightarrowtail} \cdots$$

with indecomposable $E_i \in \operatorname{add} \coprod_{n=0}^{\infty} (L^{-n}u)_-$. Then $\varinjlim E_i$ belongs to Λ-**Lat** and is indecomposable. □

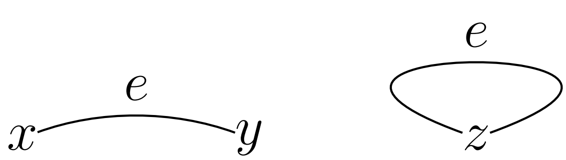
Note: $\prod \operatorname{ind}\Lambda$ is not a lattice, since $R^{\aleph_0}$ is not free!

7. The global case

A pair $(X; E)$ of finite sets ($X = \{\text{vertices}\}, E = \{\text{edges}\}$) with an incidence function $E \times X \to \mathbb{N}$, $(e, x) \mapsto ex$, is a *hypergraph* if for edges $e \in E$ and vertices $x \in X$, the *degrees* are positive:

$$d(e) := \sum_{y \in X} ey > 0\,; \quad d(x) := \sum_{f \in E} fx > 0.$$

For an ordinary graph, all edges have degree 2:



$ex = ey = 1$ $\qquad$ $ez = 2$

Let $K\Lambda$ be separable. We associate a hypergraph $\mathrm{H}(\Lambda) = (\mathfrak{X}_\Lambda, \mathcal{E}_\Lambda)$ to Λ as follows.

Vertices and edges:

$$\mathcal{X}_\Lambda := \bigcup_{\mathfrak{p}\in \mathrm{S}(\Lambda)} \operatorname{ind} A_\mathfrak{p}\,; \quad \mathcal{E}_\Lambda := \bigcup_{\mathfrak{p}\in \mathrm{S}(\Lambda)\cup\{0\}} \operatorname{ind} \Lambda_\mathfrak{p}.$$

So $\mathcal{X}_\Lambda$ consists of simple $A_\mathfrak{p}$-modules X, $\mathfrak{p} \in \mathrm{S}(\Lambda)$.
$\curvearrowright D_X := \operatorname{End}_{A_\mathfrak{p}}(X)$ is a skew-field.

Incidence: For $X \in \operatorname{ind} A_\mathfrak{p}$ and $E \in \operatorname{ind} \Lambda_\mathfrak{q}$, put

$$EX := \dim_{D_X} \operatorname{Hom}_{\Lambda\mathfrak{p}}(E, X),$$

if $\mathfrak{q} \subset \mathfrak{p}$, otherwise $EX := 0$. There are two cases:
1. *E rational* ($\mathfrak{q} = 0$): $E \in \operatorname{ind} K\Lambda$, $E_\mathfrak{p} = \bigoplus X^{EX}$
2. *E integral* ($\mathfrak{q} = \mathfrak{p}$): $E \in \operatorname{ind} \Lambda_\mathfrak{p}$, $KE = \bigoplus X^{EX}$.

The vertices of the rational edges partition $\mathcal{X}_\Lambda$.

Theorem 5. Λ *satisfies* (FD) *if and only if*
(a) $|\operatorname{ind} \Lambda_\mathfrak{p}| < \infty$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$.
(b) $\mathrm{H}(\Lambda)$ *cycle-free and "completely solvable".*

The latter combinatorial condition can be checked by successive reduction of $\mathrm{H}(\Lambda)$. Condition (a) says that Λ is *locally lattice-finite*. By Jones' theorem, this is equivalent to "lattice-finite" in case K is an algebraic number field.

Proof. We generalize the theory of genus to Λ-**Lat**, making heavy use of the local lattice-finiteness (a). A fundamental observation was made in BCK (2004):

$$\Lambda\text{-}\mathbf{Lat} \approx (\Lambda\text{-}\mathbf{lat})\text{-}\mathbf{Proj}.$$

In categorical terms, we also have:

Λ locally lattice-finite $\Leftrightarrow$ (Λ-**lat**)-**mod** noetherian.

The category Λ-**Lat** has enough projectives, and enough injectives:

$$\mathbf{Proj}(\Lambda\text{-}\mathbf{Lat}) \approx \mathbf{Inj}(\Lambda\text{-}\mathbf{Lat}) \approx \Lambda\text{-}\mathbf{Proj}.$$

Every morphism in Λ-**Lat** has a kernel, but need not have a cokernel.

Infinite rank genera. (a) implies that in Λ-**Lat**, "split epimorphism" is still a local property. Hence

Proposition 1. *Let Λ be locally lattice-finite.* $\curvearrowright$ *Projectivity of $P \in \Lambda$-***Lat** *is a local property.*

As another step toward Theorem 5, we prove that full decomposablity is an invariant of the genus:

Proposition 2. *Assume (a). Let $L \in \Lambda$-***Lat** *be in the genus of $\coprod_{i\in I} E_i$ with $E_i \in \operatorname{ind}\Lambda$. Then $L \cong \coprod_{i\in I} F_i$ with F_i in the genus of E_i.*

The combinatorial part of Theorem 5 amounts to the solution of a system S of Diophantine equations, encoded in the hypergraph $\mathrm{H}(\Lambda)$. The solutions of S correspond to representations $L \in \Lambda$-**Lat** with prescribed localizations. Condition (b) of Theorem 5 refers to the solvability of S, which can be decided from $\mathrm{H}(\Lambda)$ by means of a reduction algorithm.

8. Large projectives

As there exist representation-finite orders without (FD), one could restrict (FD) to the subcategory Λ-**Proj** of Λ-**Lat**. The following question was raised by McGovern, Puninskiĭ, and Rothmaler (Manuscr. math., to appear):

Question. Let Λ be a left noetherian ring such that the number of generators of left ideals of Λ is uniformly bounded. Is every projective left Λ-module a direct sum of finitely generated modules?

The authors expected a positive answer. We show, however, that large projectives exist for plenty of orders Λ. We restrict $\mathrm{H}(\Lambda)$ to projective edges to get the *projective hypergraph* $\mathrm{PH}(\Lambda)$.

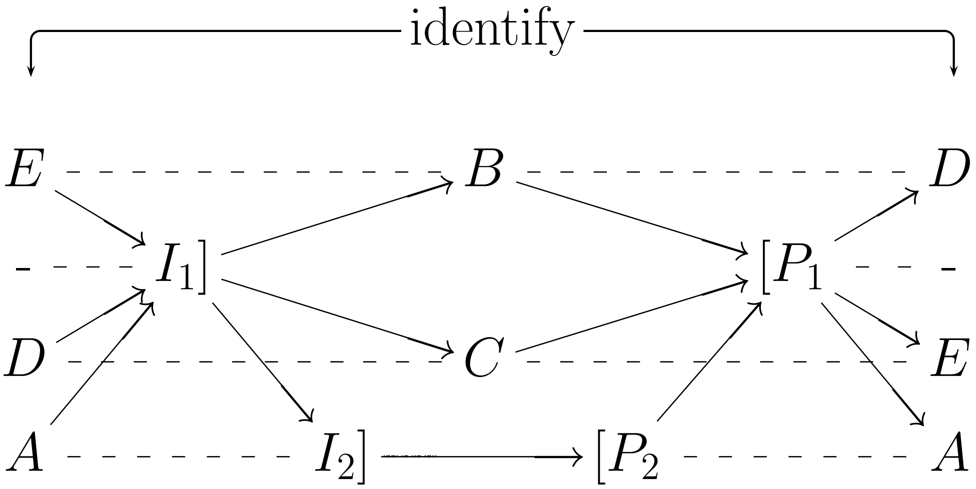
Example 5.

$$\Lambda = \mathbb{Z}\cdot 1 + \begin{pmatrix} (2) & (6) \\ \mathbb{Z} & (6) \end{pmatrix} \times \begin{pmatrix} (6) & (6) \\ \mathbb{Z} & (3) \end{pmatrix}$$

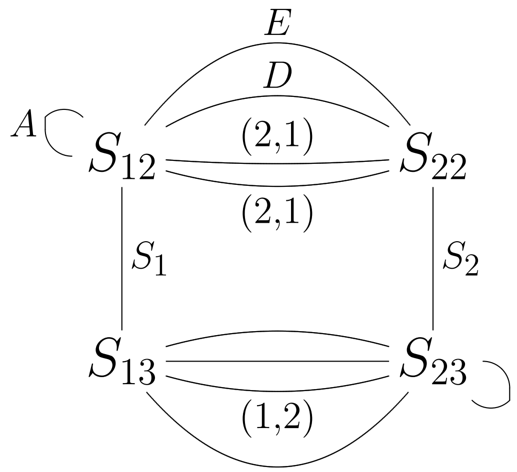
with $A = K\Lambda = \mathrm{M}_2(\mathbb{Q})\times\mathrm{M}_2(\mathbb{Q})$, $\mathrm{S}(\Lambda) = \{(2),(3)\}$. For $\mathfrak{p}\in\mathrm{S}(\Lambda)$ and $R := \mathbb{Z}_{\mathfrak{p}}$, we have

$$\Lambda_{\mathfrak{p}} \cong R\cdot 1 + \begin{pmatrix} \mathfrak{p} & \mathfrak{p} \\ R & \mathfrak{p} \end{pmatrix} \times \begin{pmatrix} \mathfrak{p} & \mathfrak{p} \\ R & R \end{pmatrix}.$$

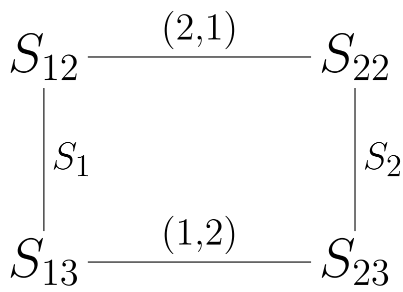
This R-order has 9 indecomposables, which make up the following Auslander-Reiten quiver:



Let S_1 and S_2 be the two simple A-modules. Their localizations give the four vertices S_{12}, S_{22}, and S_{13}, S_{23} of $\mathrm{H}(\Lambda)$. Apart from 1-loops (i. e. irreducibles) which are inessential, $\mathrm{H}(\Lambda)$ has 10 integral edges, and the two rational edges S_1, S_2.



The projective hypergraph $\mathrm{PH}(\Lambda)$ looks as follows:



This yields a large projective $P \in \Lambda$-**Lat** without finitely generated direct summands!

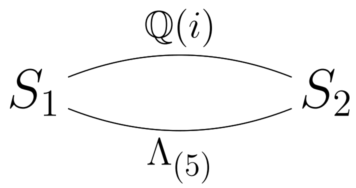
9. Further examples

An order with a loop in $\mathrm{H}(\Lambda)$ arises already for one singular prime:

Example 6. $\Lambda = \mathbb{Z} \oplus \mathbb{Z}5i$ with $i = \sqrt{-1}$. Then $\mathrm{S}(\Lambda) = \{(5)\}$. Since $(\frac{-1}{5}) = 1$, we have two vertices:

$$\mathbb{Q}_5(i) \cong \mathbb{Q}_5 \times \mathbb{Q}_5.$$

Let us denote them by S_1 and S_2. The rational edge $\mathbb{Q}(i)$ together with the $\mathbb{Z}_5$-order $\Lambda_{(5)}$ makes up a cycle:



Example 7. Finally, let Λ be the integral group ring $\mathbb{Z}C_{p^2}$ for a rational prime p. Here Λ is locally lattice-finite with $\mathrm{S}(\Lambda) = \{(p)\}$. If ζ_n denotes a primitive p^nth root of unity, we have

$$\mathbb{Q}_p\Lambda \cong \mathbb{Q}_p \times \mathbb{Q}_p(\zeta_1) \times \mathbb{Q}_p(\zeta_2),$$

which gives three vertices. However, a similar decomposition already holds for $\mathbb{Q}\Lambda$, i. e. the rational edges are 1-loops. This implies that Λ satisfies the full decomposability condition (FD).