Scientific Achievements and Prospects

My principal aim is to explore the realm of Pure Mathematics which is, by its very nature, one whole territory, with as many districts as there are different mathematical subjects. Despite of all human attempts to split mathematics into parts, incited by a more or less natural classification principle, impelled to purge a particular theory from external ideas, even if they contain the original motivation, mathematics ever resisted such particularism and brought to light its indestructible unity.

1. Classical Orders. In 1980, I started to work on classical orders and their integral representations. I classified indecomposable systems of lattices in vector spaces by invariants [3, 4] and applied this to representations of tiled orders. I observed a relationship to an old problem of Birkhoff (1935) and solved it in the representation-finite case. Recently, a new attack to Birkhoff's problem was made by Ringel and Schmidmeier (Trans. AMS, 2005).

I developed a covering technique for completely reducible orders which led to a classification of indecomposable lattice systems in the representation-finite case [5]. The complete list (4 infinite series and 854 exceptional representations) is given in [7].

For finite dimensional algebras over a field, the stability of indecomposable representations with respect to deformations was proved by Gabriel in 1974. For classical orders, the situation is quite intricate, and such a phenomenon is hard to describe in the absence of a base field. In a tour de force in 1989, I finally got a stability theorem for completely reducible orders [9]. A simplified proof is sketched in [10], but it remains to be complicated.

The global dimension of tiled orders was first investigated in the early seventies by Tarsy and Jategaonkar. In [14], I associate a cell complex to such an order and prove that the global dimension can be read off from this complex and the characteristic of the base ring.

I showed that the dependance on the characteristic actually occurs if certain homology groups of the complex are non-zero. This phenomenon remained undetected as it does not arise for orders of small rational length.

A major breakthrough in representation theory of algebras and orders was made by Auslander and Reiten in the mid-seventies. By means of their almost split sequences, the structure of categories of representations can be determined in terms of a valued graph, the Auslander-Reiten quiver. In [48], I show that almost split sequences of classical orders (i. e. one-dimensional Cohen-Macaulay orders) can be regarded as amalgamations of their end-terms. This property has no analogue for dimensions other than 1.

2. Orders in Non-Semisimple Algebras. If the ambient algebra of an order is not semisimple, the Auslander-Reiten strategy breaks down. In terms of non-commutative geometry, this drawback is caused by a non-isolated singularity. Orders of that non-classical type are investigated in [8, 18, 19, 22]. The simplest type of a non-isolated two-dimensional singularity arose in Roggenkamp's work on Green orders and Hecke algebras. In [22], I introduce a partial Auslander-Reiten quiver and obtain a complete list of indecomposable representations for this type of orders.

3. Non-Commutative Algebraic Geometry. The first step towards non-commutative algebraic geometry consists in a proper concept of regularity. For classical orders, such a generalization is easily given. It simply states that the global dimension is equal to that of the base ring. If finiteness over a base ring is dropped, Auslander-regularity and Macaulayness seems to be a good regularity concept. For regular algebras in the sense of Artin and Schelter, generalized regular sequences consisting of invertible ideals play a rôle, but invertible ideals definitely fail to decribe the most general type of regularity. In [24], I use invertible ideal sequences to introduce a class of regular rings which can be described in terms of higher dimensional discrete valuations, with values in a completed l-group. A similar class of non-commutative regular rings which comprise localized enveloping algebras of finite dimensional Lie algebras is considered in [34]. Classical orders without a base ring are studied in [30].

Much of the classical theory of orders survives if the base ring is dropped. From a geometric point of view, removal of the base ring appears to be natural and even essential for a study of non-commutative behaviour. In [31, 33, 35], and [44], some steps of this project are carried out.

A very important tool for the structural analysis of categories of representations was introduced in its rudiments by Igusa and Todorov in 1984. Their "ladders" enabled them to give a homological characterization of finite Auslander-Reiten quivers of artinian algebras. Around 2000, Iyama improved this method considerably. In his three papers on τ-categories (published 2005), he obtained a similar result for classical orders. Based on his achievements, I introduced L-functors (=ladder functors) [45] to get a flexible instrument which led to a proof of his conjecture that finite Auslander-Reiten quivers of classical orders can be characterized in terms of additive functions. In [32], I applied L-functors to artinian algebras, and thereby removed the technicalities in the proof of Igusa and Todorov's result. In [38, 40], and [43], the theory of L-functors is developed further, and it is shown that L-functors also apply to Cohen-Macaulay orders beyond the critical dimension 2, where τ-categories no longer exist.

4. Two-point Differentiation. For tiled orders on the one hand, and finite partially ordered sets on the other hand, Zavadskij developed a peculiar algorithm to reduce a representation-finite tiled order or a poset within finitely many steps to a trivial one. Simson generalized this algorithm to Schurian vector space categories. Because of its combinatorial nature, the algebraic essence of the method remained undetected. In [21], I obtained a purely module-theoretic description of Zavadskij's algorithm. In this way, a generalization to arbitrary orders became possible. A further generalization and its relationship to Auslander-Reiten theory is given in [25, 23]. The ultimate generalization to quasi-abelian categories, and its application to artinian algebras, is presented in [37].

5. Quasi-abelian Categories and Tilting. Apart from constituting a proper domain for differentiation, quasi-abelian categories arise as categories of representations for artinian algebras, classical orders, and even two-dimensional Cohen-Macaulay orders, where Auslander-Reiten sequences exist without exceptions at projectives or injectives. In the systematic treatment [28], I call them 'almost abelian'. Being close to abelian categories, they interrelate quite a number of seemingly different structures to each other. They bear the categorical essence of tilting theory [28, 29], as they always give rise to a tilting adjunction between abelian categories. Among other things, quasi-abelian categories are intimately related to torsion theories and provide a deeper understanding of Pontrjagin duality.

6. McKay Correspondence. Over an algebraically closed field of characteristic zero, every cocommutative Hopf algebra can be represented as a twisted group ring, where the group G operates on the universal enveloping algebra of a Lie algebra. If the base field is uncountable, and the Lie algebra L is finite dimensional and solvable, the Dixmier correspondence relates orbits on L^* under the algebraic group $G(L)$ to primitive ideals of $U(L)$, while G is trivial. In [17], I investigate the complementary case, where L is abelian and G finite. This leads to a similar bijection between G-orbits on L^* and irreducible $U[G]$-modules. A quantized version of this correspondence was studied by Crawley-Boevey and Holland. There is a close relationship to preprojective algebras in the tame case. The representation-finite case is treated in [16]. Here the hereditary algebra A of a Dynkin diagram is deformed into a semisimple algebra $\widetilde{A}$, so that the indecomposable representations of A correspond to the simple modules over $\widetilde{A}$.

7. Large lattices. Integral representations of infinite rank over C_2 occurred, almost at the same time, in the context of Lie algebras (Bryant, 2000) and C^*-algebras (Kumjian and Phillips, 2002). Butler, Campbell, and Kovács replaced C_2 by a cyclic group of prime order and proved that the classical theory of Diederichsen and Reiner carries over to that case. This marked the beginning of a theory of integral representations of infinite rank. While the local case could be treated fairly analoguously to large modules over artinian algebras, except that at one place, I had to make use of L-functors [39], quite unexpected phenomena arose in the global theory [42]. Here a new theory of genera had to be developed, which showed that the class of orders where the classical theory remains true is rather narrow. The criterion [42] is given in terms of a hypergraph associated to the given order. It decides whether each infinite rank representation decomposes into those of finite rank, by a combinatorial reduction in finitely many steps. For example, the criterion shows that group rings over p-groups of nilpotency class 2 behave classically. Further results are obtained in [46, 53].

8. Braces, and the Quantum Yang-Baxter Equation. On the ICM 1990, Drinfeld initiated the study of set-theoretic solutions of the quantum Yang-Baxter equation, which cannot be obtained by deformations of the trivial solution. Etingof, Schedler, and Soloviev [ESS], and independently, Gateva-Ivanova and Van den Bergh, investigated such solutions which arise, for example, from a certain class of Artin-Schelter regular rings of arbitrary finite global dimension. These solutions are called "square-free". The mentioned authors conjectured that every square-free solution comes from such an Artin-Schelter regular ring. [ESS] related the conjecture to a decomposablitiy property which every square-free solution was supposed to have. In [41], I prove this conjecture. In the same paper, I introduce cycle sets, which describe a deformation of free abelian groups into non-commutative groups. Similar structures arose in connection with Sklyanin algebras (Tate, Van den Bergh, 1996) and quantum groups (Etingof, Gelaki, 1998). By a further analysis, I arrived at the concept of brace [51], an additive version of a cycle set. Every radical ring can be regarded as a brace, and for this reason, a new and stronger conjecture on set-theoretic solutions of the QYBE can be viewed as a nilpotency problem for generalized radical rings, i. e. braces. There is a module theory over braces [56], and in a sense, braces are even more fundamental than radical rings. Another open question at the end of [ESS] is answered in [57].

9. Vector Space Categories, and Quasi-crystals. Vector space categories were introduced by Nazarova and Roiter in order to prove the second Brauer-Thrall conjecture. Around 1980, Ringel established their importance for representation theory of artinian algebras. Klemp and Simson classified critical Schurian vector space categories, and thereby extended previous results of Kleiner and Nazarova on representations of partially ordered sets. Generalizing Roiter's norm of a finite poset, I introduce a norm [50] for arbitrary Schurian vector space categories $\mathcal{C}$, such that $\mathcal{C}$ is representation-finite if and only if it has finitely many isomorphism classes of indecomposable objects and its norm is greater than 1/4. Recently, Nazarova and Roiter introduced the concept of P-faithful poset and showed that the P-faithful posets of norm 1/4 coincide with Kleiner's list of critical posets. Their conjecture on the precise form of P-faithful posets was established by Zeldich and Sapelkin, with a huge amount of combinatorial reductions. Using Auslander-Reiten theory, I obtain a natural proof in [50], relating P-faithful posets to hereditary algebras with a particularly nice Auslander-Reiten quiver. A relationship to quasi-crystals is pursued in [47] and [54].

10. Abelian l-groups. Since 2004, I became interested in the vast theory of lattice-ordered groups which interacts with rather different mathematical theories like group and ring theory, logic and model theory, functional analysis, topology, universal algebra and lattice theory. Its origin can be traced back to the year 1901, when Otto Hölder proved the embeddability of archimedean ordered groups into the real line. Hahn constructed a class of abelian ordered groups which comprise every abelian ordered group as a subgroup. Ordered groups also occur in Hilbert's work on the foundation of classical geometry. Abelian l-groups first arise as topological vector spaces in Hilbert's theory of integral equations. They have been investigated further in papers of Riesz, Freudenthal, Kantorovich, Artin, Schreier, Birkhoff, Stone, and Yoshida. The study of l-groups in their own right began with the work of Birkhoff and Nakano. Levi proved that every torsion-free abelian group can be made into an abelian l-group. For the most part, the present shape of abelian l-group theory is due to the pioneering work of Paul Conrad who proved a number of deep structure theorems. His legendary "blue notes" (Tulane Lecture notes, 1970) inspired intensive research on the topic and created an essential part of the monograph of Bigard, Keimel, and Wolfenstein (1977).

There is a triangle correspondence between abelian l-groups, Bézout domains, and MV-algebras (related to many-valued logic). For a given Bézout domain R with quotient field K, the group $G(R) := K^\times/R^\times$ carries a natural structure of an abelian l-group. The converse is more recondite. It was established by Jaffard, Ohm, and Kaplansky. In his expository lecture on the Curaçao conference in 1988, M. Anderson conjectured that every l-embedding $G(R) \hookrightarrow H$ arises from an extension $R \hookrightarrow S$ of Bézout domains. We prove this in [62]. Abelian l-groups also shed some light upon commutative ring theory. In [60], I take up former investigations of Popescu and Vraciu of 1976, and obtain new results on sheaves of field extensions. The paper [59] (with Y. C. Yang) contains a revision of Bernau's embedding theorem. We construct the lateral completion of an archimedean l-group G directly from the structure sheaf of G. It turns out that to a large extent, the passage from G to its essential closure merely depends on topological operations on spectral spaces. Topological considerations play a decisive rôle in [63, 64]. Here we give a categorical analysis of the absolute $P \to X$ of an arbitrary topological space X. (For regular spaces, this concept is equivalent to a projective cover.) As an application [63], the strongly projectable hull of an abelian l-group G is obtained as a unique lifting of the absolute $P \to X := \operatorname{Spec} G$.

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