Idempotent generation in partition monoids

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Joint work with Bob Gray (and others)
Shona says Hi . . .
Partition Monoids

Let \( n = \{1, \ldots, n\} \) and \( n' = \{1', \ldots, n'\} \).

The partition monoid on \( n \) is \( P_n = \{\text{set partitions of } n \cup n'\} \equiv \{\text{graphs on vertex set } n \cup n'\} \).

Eg: \( \alpha = \{\},\ldots,\{\} \in P_6 \)

Note: \( P_n \) is the basis of the partition algebra \( P_\delta n \).

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Idempotent generation in partition monoids
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Note: $P_n$ is the basis of the partition algebra $P_{\delta n}$. 

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Idempotent generation in partition monoids
Let $\mathbf{n} = \{1, \ldots, n\}$ and $\mathbf{n'} = \{1', \ldots, n'\}$.

\begin{align*}
\mathbf{n} &= \{1, 2, 3, 4, 5, 6\} \\
\mathbf{n'} &= \{1', 2', 3', 4', 5', 6'\}
\end{align*}
Let \( n = \{1, \ldots, n\} \) and \( n' = \{1', \ldots, n'\} \).

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\[ \mathcal{P}_n = \{\text{set partitions of } n \cup n'\} \]

Eg: \( \alpha = \left\{ \{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 6'\}, \{2', 3'\}, \{5'\} \right\} \in \mathcal{P}_6 \)

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

\[
\begin{array}{cccccc}
1' & 2' & 3' & 4' & 5' & 6' \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

Note: \( \mathcal{P}_n \) is the basis of the partition algebra \( \mathcal{P}_\delta \).

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Partition Monoids

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  \[ \equiv \{\text{graphs on vertex set } n \cup n'\}. \]
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1' & 2' & 3' & 4' & 5' & 6' \\
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\mathcal{P}_n = \{ \text{set partitions of } n \cup n' \} = \{ \text{graphs on vertex set } n \cup n' \}.
\]

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Note: \( \mathcal{P}_n \) is the basis of the partition algebra \( \mathcal{P}_{\delta_n} \).
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\[1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6\]

\[\Rightarrow n\]

\[1' \quad 2' \quad 3' \quad 4' \quad 5' \quad 6'\]

\[\Rightarrow n'\]
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- Note: \( \mathcal{P}_n \) is the basis of the partition algebra \( \mathcal{P}^\delta_n \).
Product in $\mathcal{P}_n$

Let $\alpha, \beta \in \mathcal{P}_n$.

To calculate $\alpha \beta$:

1. connect bottom of $\alpha$ to top of $\beta$,
2. remove middle vertices and floating components,
3. smooth out resulting graph to obtain $\alpha \beta$.

The operation is associative, so $\mathcal{P}_n$ is a semigroup (monoid, etc).

Note: usual multiplication in partition algebra $\mathcal{P}_\delta$ with $\delta = 1$. 

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Let $\alpha, \beta \in \mathcal{P}_n$. To calculate $\alpha \beta$:

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(2) *remove middle vertices and floating components*,

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Idempotent generation in partition monoids
Submonoids of $P_n$

* $B_n = \{ \alpha \in P_n : |A| = 2 \ (\forall A \in \alpha) \}$ — Brauer monoid $\in B_5$

* $TL_n = \{ \alpha \in B_n : \alpha \text{ is planar} \}$ — Temperley-Lieb monoid $\in TL_5$ (aka Jones or Kauffman monoid)

* $S_n = \{ \alpha \in B_n : |A \cap n| = |A \cap n'| = 1 \ (\forall A \in \alpha) \}$ — symmetric group $\in S_5$
Submonoids of $\mathcal{P}_n$

- $\mathcal{B}_n = \{ \alpha \in \mathcal{P}_n : |A| = 2 (\forall A \in \alpha) \}$ — Brauer monoid

$$\in \mathcal{B}_5$$
Submonoids of $\mathcal{P}_n$

- $\mathcal{B}_n = \{ \alpha \in \mathcal{P}_n : |A| = 2 \ (\forall A \in \alpha) \}$  — Brauer monoid

- $\mathcal{T\mathcal{L}}_n = \{ \alpha \in \mathcal{B}_n : \alpha \text{ is planar} \}$  — Temperley-Lieb monoid
  
  (aka Jones or Kauffman monoid)

  \[
  \begin{array}{c}
  \begin{picture}(20,20)
    \put(-10,10){\circle*{3}}
    \put(-10,-10){\circle*{3}}
    \put(10,10){\circle*{3}}
    \put(10,-10){\circle*{3}}
    \put(-5,5){\line(1,0){20}}
    \put(-5,-5){\line(1,0){20}}
    \put(-5,5){\line(1,1){10}}
    \put(-5,-5){\line(1,-1){10}}
  \end{picture}
  \\
  \in \mathcal{B}_5
  
  \begin{picture}(20,20)
    \put(-10,10){\circle*{3}}
    \put(-10,-10){\circle*{3}}
    \put(10,10){\circle*{3}}
    \put(10,-10){\circle*{3}}
    \put(-5,5){\line(1,0){20}}
    \put(-5,-5){\line(1,0){20}}
    \put(-5,5){\line(1,1){10}}
    \put(-5,-5){\line(1,-1){10}}
  \end{picture}
  \\
  \in \mathcal{T\mathcal{L}}_5
  \end{array}
  \]
Submonoids of \( \mathcal{P}_n \)

- \( \mathcal{B}_n = \{ \alpha \in \mathcal{P}_n : |A| = 2 \ (\forall A \in \alpha) \} \quad — \text{Brauer monoid} \\

- \( \mathcal{TL}_n = \{ \alpha \in \mathcal{B}_n : \alpha \text{ is planar} \} \quad — \text{Temperley-Lieb monoid (aka Jones or Kauffman monoid)} \\

- \( \mathcal{S}_n = \{ \alpha \in \mathcal{B}_n : |A \cap \mathbf{n}| = |A \cap \mathbf{n}'| = 1 \ (\forall A \in \alpha) \} \quad — \text{symmetric group} \\

In the diagram:

- \( \mathcal{B}_5 \)
- \( \mathcal{TL}_5 \)
Submonoids of $\mathcal{P}_n$

\[ \mathcal{T}_n = \{ \alpha \in \mathcal{P}_n : |A \cap n'| = 1 \ (\forall A \in \alpha) \} \]

— full transformation semigroup

\[ \in \mathcal{T}_5 \]
Submonoids of $\mathcal{P}_n$

- $\mathcal{T}_n = \{ \alpha \in \mathcal{P}_n : |A \cap n'| = 1 \ (\forall A \in \alpha) \}$
  
  — full transformation semigroup

- $\mathcal{T}_n^* = \{ \alpha \in \mathcal{P}_n : |A \cap n| = 1 \ (\forall A \in \alpha) \}$
  
  — $\mathcal{T}_n^* \cong^{\text{op}} \mathcal{T}_n$
Submonoids of $\mathcal{P}_n$

- $\mathcal{T}_n = \{ \alpha \in \mathcal{P}_n : |A \cap n'| = 1 \ (\forall A \in \alpha) \}$
  - full transformation semigroup

- $\mathcal{T}_n^* = \{ \alpha \in \mathcal{P}_n : |A \cap n| = 1 \ (\forall A \in \alpha) \}$
  - $\mathcal{T}_n^* \cong^{op} \mathcal{T}_n$

- $\mathcal{I}_n = \{ \alpha \in \mathcal{P}_n : |A \cap n'| \leq 1 \text{ and } |A \cap n| \leq 1 \ (\forall A \in \alpha) \}$
  - symmetric inverse monoid (aka rook monoid)
Submonoids of $\mathcal{P}_n$

- $\mathcal{T}_n = \{ \alpha \in \mathcal{P}_n : |A \cap n'| = 1 \ (\forall A \in \alpha) \}$ — full transformation semigroup

$$\in \mathcal{T}_5$$

- $\mathcal{T}_n^* = \{ \alpha \in \mathcal{P}_n : |A \cap n| = 1 \ (\forall A \in \alpha) \}$ — $\mathcal{T}_n^* \cong^{\text{op}} \mathcal{T}_n$

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$$\in \mathcal{I}_5$$

- In many ways, $\mathcal{P}_n$ is just like a transformation semigroup.
Proposition

There is a factorization $\mathcal{P}_n = \mathcal{T}_n \mathcal{I}_n \mathcal{T}_n^*$. Consequently,

$$\mathcal{P}_n = \langle s_1, \ldots, s_{n-1}, e_1, \ldots, e_n, t_1, \ldots, t_{n-1} \rangle.$$
The partition monoid $\mathcal{P}_n$ has presentation

$$\mathcal{P}_n \cong \langle s_1, \ldots, s_{n-1}, e_1, \ldots, e_n, t_1, \ldots, t_{n-1} : (R1—R16) \rangle,$$

where

(R1) $s_i^2 = 1$  \hspace{1cm} (R9) $t_i^2 = t_i$
(R2) $s_is_j = s_js_i$  \hspace{1cm} (R10) $t_it_j = t_jt_i$
(R3) $s_is_js_i = s_js_is_j$  \hspace{1cm} (R11) $s_it_j = t_js_i$
(R4) $e_i^2 = e_i$  \hspace{1cm} (R12) $s_isjt_i = t_js_is_j$
(R5) $e_i e_j = e_j e_i$  \hspace{1cm} (R13) $t_is_i = s_it_i = t_i$
(R6) $s_ie_j = e_js_i$  \hspace{1cm} (R14) $t_ie_j = e_jt_i$
(R7) $s_ie_i = e_{i+1}s_i$  \hspace{1cm} (R15) $t_ie_jt_i = t_i$
(R8) $e_ie_{i+1}s_i = e_ie_{i+1}$  \hspace{1cm} (R16) $e_jt_ie_j = e_j$. 

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Idempotent generation in partition monoids
The partition algebra $\mathcal{P}_n^\delta$ has presentation

$$
\mathcal{P}_n^\delta \cong \langle s_1, \ldots, s_{n-1}, e_1, \ldots, e_n, t_1, \ldots, t_{n-1} : (R1—R16) \rangle,
$$

where

(R1) $s_i^2 = 1$

(R2) $s_is_j = s_js_i$

(R3) $s_is_js_i = s_js_is_j$

(R4) $e_i^2 = \delta e_i$

(R5) $e_i e_j = e_j e_i$

(R6) $s_i e_j = e_j s_i$

(R7) $s_i e_i = e_{i+1} s_i$

(R8) $e_i e_{i+1} s_i = e_i e_{i+1}$

(R9) $t_i^2 = t_i$

(R10) $t_i t_j = t_j t_i$

(R11) $s_i t_j = t_j s_i$

(R12) $s_i s_j t_i = t_j s_i s_j$

(R13) $t_i s_i = s_i t_i = t_i$

(R14) $t_i e_j = e_j t_i$

(R15) $t_i e_j t_i = t_i$

(R16) $e_j t_i e_j = e_j$. 

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Idempotent generation in partition monoids
Theorem (E, 2011)

The singular partition monoid $\mathcal{P}_n \setminus S_n$ has presentation

$$\mathcal{P}_n \setminus S_n \cong \langle e_1, \ldots, e_n, t_{ij} \ (1 \leq i < j \leq n) : (\text{R1—R10}) \rangle,$$

where

- (R1) $e_i^2 = e_i$
- (R2) $e_i e_j = e_j e_i$
- (R3) $t_{ij}^2 = t_{ij}$
- (R4) $t_{ij} t_{kl} = t_{kl} t_{ij}$
- (R5) $t_{ij} e_k t_{ij} = t_{ij}$
- (R6) $e_k t_{ij} e_k = e_k$
- (R7) $t_{ij} e_k = e_k t_{ij}$
- (R8) $t_{ij} t_{jk} = t_{jk} t_{ki} = t_{ki} t_{ij}$
- (R9) $e_k t_{ki} e_i t_{ij} e_j t_{jk} e_k = e_k t_{kj} e_j t_{ji} e_i t_{ik} e_k$
- (R10) $e_k t_{ki} e_i t_{ij} e_j t_{jl} e_l t_{lk} e_k = e_k t_{kl} e_l t_{li} e_i t_{ij} e_j t_{jk} e_k$.

$$e_i = \begin{array}{cccc} 1 & \cdots & i & \cdots & n \end{array}$$

$$t_{ij} = \begin{array}{cccc} 1 & \cdots & i & \cdots & j & \cdots & n \end{array}$$
The Brauer monoid $\mathcal{B}_n$ has presentation

$$\mathcal{B}_n \cong \langle s_1, \ldots, s_{n-1}, u_1, \ldots, u_{n-1} : (R1—R10) \rangle,$$

where

- (R1) $s_i^2 = 1$
- (R2) $s_is_j = s_ks_i$
- (R3) $s_is_js_i = s_ks_ks_i$
- (R4) $u_i^2 = u_i$
- (R5) $u_iu_j = u_ju_i$
- (R6) $s_iu_j = u_js_i$
- (R7) $s_iu_i = u_is_i = u_i$
- (R8) $u_iu_ju_i = u_i$
- (R9) $s_iu_ju_i = s_ju_i$
- (R10) $u_iu_js_i = u_is_j$. 

Diagrams:

- $s_i = \begin{array}{cccc}
\cdots & & & \\
\cdot & & \times & \\
\cdot & & & \cdots
\end{array}$
- $u_i = \begin{array}{cccc}
\cdots & & & \\
\cdot & & \circ & \\
\cdot & & & \cdots
\end{array}$
The singular Brauer monoid $\mathcal{B}_n \setminus \mathcal{S}_n$ has presentation

$$\mathcal{B}_n \setminus \mathcal{S}_n \cong \langle u_{ij} \mid 1 \leq i < j \leq n \rangle : (R1—R6),$$

where

(R1) $u_{ij}^2 = u_{ij}$

(R2) $u_{ij} u_{kl} = u_{kl} u_{ij}$

(R3) $u_{ij} u_{jk} u_{ij} = u_{ij}$

(R4) $u_{ij} u_{ik} u_{jk} = u_{ij} u_{jk}$

(R5) $u_{ij} u_{jk} u_{kl} = u_{ij} u_{il} u_{kl}$

(R6) $u_{ij} u_{kl} u_{ik} = u_{ij} u_{jl} u_{ik}$. 

$u_{ij} =$

\begin{center}
\begin{tikzpicture}

\node (1) at (0,0) [circle,fill,inner sep=2pt]{1};
\node (i) at (1,0) [circle,fill,inner sep=2pt]{};
\node (j) at (2,0) [circle,fill,inner sep=2pt]{};
\node (n) at (3,0) [circle,fill,inner sep=2pt]{};

\draw (1) -- (i);
\draw (i) -- (j);
\draw (j) -- (n);
\draw (i) -- (j);
\end{tikzpicture}
\end{center}
The (singular) Temperley-Lieb monoid $\mathcal{TL}_n$ has presentation

$$\mathcal{TL}_n \cong \langle u_1, \ldots, u_{n-1} : (R1—R3) \rangle,$$

where

(R1) $u_i^2 = u_i$  \hspace{1cm} (R2) $u_i u_j = u_j u_i$ \hspace{1cm} (R3) $u_i u_j u_i = u_i$.

\begin{center}
\begin{tikzpicture}
    \node (1) at (0,0) {$1$};
    \node (n) at (3,0) {$n$};
    \node (i) at (1.5,0) {$i$};
    \node[] (mid) at (1.5,-0.5) {};\node[dot] (mid) at (1.5,-0.5) {};
    \node[] (end) at (1.5,-1) {};\node[dot] (end) at (1.5,-1) {};
    \node[] (start) at (0,-1) {};\node[dot] (start) at (0,-1) {};
    \node[] (end2) at (3,-1) {};\node[dot] (end2) at (3,-1) {};
    \draw (1) -- (mid);\draw (mid) -- (n);
    \draw (start) -- (mid) -- (end);
    \draw (mid) to [out=45,in=-45] (mid);\end{tikzpicture}
\end{center}

\[ u_i = \]

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Idempotent generation in partition monoids
So the singular parts of $P_n, B_n, TL_n$ are idempotent generated . . .
So the singular parts of $\mathcal{P}_n$, $\mathcal{B}_n$, $\mathcal{T\mathcal{L}}_n$ are idempotent generated . . .

- What is the smallest number of (idempotent) partitions required to generate $\mathcal{P}_n \setminus \mathcal{S}_n$?
So the singular parts of $\mathcal{P}_n$, $\mathcal{B}_n$, $\mathcal{T\mathcal{L}_n}$ are idempotent generated . . .

- What is the smallest number of (idempotent) partitions required to generate $\mathcal{P}_n \setminus S_n$?

- i.e., What is the rank and idempotent rank, $\text{rank}(\mathcal{P}_n \setminus S_n)$ and $\text{idrank}(\mathcal{P}_n \setminus S_n)$?
Idempotent generation — questions

So the singular parts of $\mathcal{P}_n$, $\mathcal{B}_n$, $\mathcal{T}\mathcal{L}_n$ are idempotent generated . . .

- What is the smallest number of (idempotent) partitions required to generate $\mathcal{P}_n \setminus S_n$?

- i.e., What is the rank and idempotent rank, $\text{rank}(\mathcal{P}_n \setminus S_n)$ and $\text{idrank}(\mathcal{P}_n \setminus S_n)$?

- How many (idempotent) generating sets of minimal size are there?

What about other ideals of $\mathcal{P}_n$?

What about infinite partition monoids $\mathcal{P}_X$?

Same questions for $\mathcal{B}_n$ and $\mathcal{T}\mathcal{L}_n$. . .
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- How many (idempotent) generating sets of minimal size are there?

- What about other ideals of $\mathcal{P}_n$?

- How many idempotents does $\mathcal{P}_n$ contain?
So the singular parts of $P_n$, $B_n$, $TL_n$ are idempotent generated . . .

- What is the smallest number of (idempotent) partitions required to generate $P_n \setminus S_n$?

- i.e., What is the rank and idempotent rank, $\text{rank}(P_n \setminus S_n)$ and $\text{idrank}(P_n \setminus S_n)$?

- How many (idempotent) generating sets of minimal size are there?

- What about other ideals of $P_n$?

- How many idempotents does $P_n$ contain?

- What about infinite partition monoids $P_\mathbb{X}$?
So the singular parts of $P_n$, $B_n$, $TL_n$ are idempotent generated . . .

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- How many idempotents does $P_n$ contain?
- What about infinite partition monoids $P_X$?
- Same questions for $B_n$ and $TL_n$ . . .
The number of idempotents in the Brauer monoid $\mathcal{B}_n$ is equal to

$$e_n = \sum_{\mu \vdash n} \frac{n!}{\mu_1! \cdots \mu_n! \cdot 2^{\mu_2} \cdots (2k)^{\mu_{2k}}}$$

where $k = \lfloor n/2 \rfloor$ — i.e., $n = 2k$ or $2k + 1$. 

Theorem (Dolinka, E, Evangelou, FitzGerald, Ham, Hyde, Loughlin, 2014)
The number of idempotents in the Brauer monoid $B_n$ is equal to

$$e_n = \sum_{\mu \vdash n} \frac{n!}{\mu_1! \cdots \mu_n! \cdot 2^{\mu_2} \cdots (2k)^{\mu_{2k}}}$$

where $k = \lfloor n/2 \rfloor$ — i.e., $n = 2k$ or $2k + 1$. The numbers $e_n$ satisfy the recurrence

- $e_0 = 1$,
- $e_n = a_1 e_{n-1} + a_2 e_{n-2} + \cdots + a_n e_0$

where $a_{2i} = \binom{n-1}{2i-1} (2i - 1)!$ and $a_{2i+1} = \binom{n-1}{2i} (2i + 1)!$. 

James East
Idempotent generation in partition monoids
Theorem (DEEFHHL, 2014)

The number of idempotent basis elements in the Brauer algebra $B_n^\delta$ is equal to

$$\sum_{\mu} \frac{n!}{\mu_1! \mu_3! \cdots \mu_{2k+1}!},$$

where

- $k = \left\lfloor \frac{n-1}{2} \right\rfloor$,
- the sum is over all integer partitions $\mu \vdash n$ with only odd parts,
- $\delta$ is not a root of unity.
The number of idempotents in the partition monoid $\mathcal{P}_n$ is equal to

$$n! \cdot \sum_{\mu \vdash n} \frac{c(1)^{\mu_1} \cdots c(n)^{\mu_n}}{\mu_1! \cdots \mu_n! \cdot (1!)^{\mu_1} \cdots (n!)^{\mu_n}},$$

where

$$c(k) = \sum_{r,s=1}^{k} (1 + rs) c(k, r, s),$$

and

- $c(k, r, 1) = S(k, r)$
- $c(k, 1, s) = S(k, s)$
- $c(k, r, s) = s \cdot c(k - 1, r - 1, s) + r \cdot c(k - 1, r, s - 1) + rs \cdot c(k - 1, r, s)$

$$+ \sum_{m=1}^{k-2} \binom{k-2}{m} \sum_{a=1}^{r-1} \sum_{b=1}^{s-1} (a(s - b) + b(r - a)) c(m, a, b) c(k - m - 1, r - a, s - b)$$

if $r, s \geq 2$. 

Theorem (DEEFHHL, 2014)

James East

Idempotent generation in partition monoids
Theorem (DEEFHHL, 2014)

The number of idempotent basis elements in the partition algebra $\mathcal{P}_n^\delta$ is equal to

$$n! \cdot \sum_{\mu \vdash n} \frac{c'(1)^{\mu_1} \cdots c'(n)^{\mu_n}}{\mu_1! \cdots \mu_n! \cdot (1!)^{\mu_1} \cdots (n!)^{\mu_n}},$$

where

- $c'(k) = \sum_{r,s=1}^{k} rs \cdot c(k, r, s)$, and
- $\delta$ is not a root of unity.
Less algebra, more diagrams...
The number of idempotents in $\mathcal{T}_n$ is currently unknown.
The number of idempotents in $\mathcal{T}L_n$ is currently unknown.
The number of idempotents in $\mathcal{T}L_n$ is currently unknown.

Thanks to Attila Egri-Nagy for these . . .
Theorem (E, 2011) \( P_n \setminus S_n \) is idempotent generated. 

\[ P_n \setminus S_n = \langle e_1, \ldots, e_n, t_{ij} (1 \leq i < j \leq n) \rangle. \]

\[ r_1 \] 

\[ \text{rank}(P_n \setminus S_n) = \text{idrank}(P_n \setminus S_n) = n + \left( \frac{n^2}{2} \right) = \left( \frac{n+1}{2} \right)^2 = n \left( \frac{n+1}{2} \right). \]
Theorem (E, 2011)

- $\mathcal{P}_n \setminus S_n$ is idempotent generated.
- $\mathcal{P}_n \setminus S_n = \langle e_1, \ldots, e_n, t_{ij} \mid 1 \leq i < j \leq n \rangle$.

$$e_r = \begin{array}{ccccccccccc}
1 & \cdots & r & \cdots & n \\
\text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots}
\end{array} \quad \quad \quad \quad
t_{ij} = \begin{array}{ccccccccccc}
1 & \cdots & i & \cdots & j & \cdots & n \\
\text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots} & \text{\ldots}
\end{array}$$
Theorem (E, 2011)

- \( \mathcal{P}_n \setminus \mathcal{S}_n \) is idempotent generated.
- \( \mathcal{P}_n \setminus \mathcal{S}_n = \langle e_1, \ldots, e_n, t_{ij} \ (1 \leq i < j \leq n) \rangle \).

- \( \text{rank}(\mathcal{P}_n \setminus \mathcal{S}_n) = \text{idrank}(\mathcal{P}_n \setminus \mathcal{S}_n) = n + \binom{n}{2} = \binom{n+1}{2} = \frac{n(n+1)}{2} \).
Any minimal idempotent generating set for $\mathcal{P}_n \setminus \mathcal{S}_n$ is a subset of

$$\{ e_r : 1 \leq r \leq n \} \cup \{ t_{ij}, f_{ij}, f_{ji}, g_{ij}, g_{ji} : 1 \leq i < j \leq n \}.$$
Any minimal idempotent generating set for $\mathcal{P}_n \setminus \mathcal{S}_n$ is a subset of

$$\{e_r : 1 \leq r \leq n\} \cup \{t_{ij}, f_{ij}, f_{ji}, g_{ij}, g_{ji} : 1 \leq i < j \leq n\}.$$ 

To see which subsets generate $\mathcal{P}_n \setminus \mathcal{S}_n$, we create a graph...
Let $\Gamma_n$ be the di-graph with vertex set

$$V(\Gamma_n) = \{A \subseteq \mathbb{n} : |A| = 1 \text{ or } |A| = 2\}$$

and edge set

$$E(\Gamma_n) = \{(A, B) : A \subseteq B \text{ or } B \subseteq A\}.$$
Minimal idempotent generating sets — $\mathcal{P}_n \setminus \mathcal{S}_n$

For only $59.95\ldots$
Let $\Gamma_n$ be the di-graph with vertex set

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$\Gamma_5$ (with loops omitted)
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A subgraph $H$ of a di-graph $G$ is a permutation subgraph if $V(H) = V(G)$ and the edges of $H$ induce a permutation of $V(G)$.
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![Diagonal graph with labeled vertices and edges](image-url)
A subgraph $H$ of a di-graph $G$ is a permutation subgraph if $V(H) = V(G)$ and the edges of $H$ induce a permutation of $V(G)$.

A permutation subgraph of $\Gamma_n$ is determined by:

- a permutation of a subset $A$ of $n$ with no fixed points or 2-cycles ($A = \{2, 3, 5\}$, $2 \mapsto 3$, $3 \mapsto 5$, $5 \mapsto 2$), and
- a function $n \setminus A \to n$ with no 2-cycles ($1 \mapsto 4$, $4 \mapsto 4$).
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![Diagram](image)
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![Diagram of a permutation subgraph](image)

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- A permutation of a subset $A$ of $\{1, 2, 3, 4, 5\}$ with no fixed points or 2-cycles ($A = \{2, 3, 5\}$, $2 \mapsto 3$, $3 \mapsto 5$, $5 \mapsto 2$), and
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- a function $\pi : \{1, \ldots, n\} \setminus A \to \{1, \ldots, n\}$ with no 2-cycles ($1 \mapsto 4$, $4 \mapsto 4$).
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- a function $\mathbf{n} \setminus A \to \mathbf{n}$ with no 2-cycles ($1 \mapsto 4$, $4 \mapsto 4$).
Theorem (E+Gray, 2014)

The minimal idempotent generating sets of $\mathcal{P}_n \setminus S_n$ are in one-one correspondence with the permutation subgraphs of $\Gamma_n$.

The number of minimal idempotent generating sets of $\mathcal{P}_n \setminus S_n$ is equal to

$$
\sum_{k=0}^{n} \binom{n}{k} a_k b_{n,n-k},
$$

where $a_0 = 1$, $a_1 = a_2 = 0$, $a_{k+1} = ka_k + k(k-1)a_{k-2}$, and

$$
b_{n,k} = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \binom{k}{2i} (2i - 1)!! n^{k-2i}.
$$

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<th>0</th>
<th>1</th>
<th>2</th>
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<th>6</th>
<th>7</th>
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<td>40915</td>
<td>754368</td>
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</table>
The ideals of \( P_n \setminus S_n \) are
\[
I_r = \{ \alpha \in P_n \setminus S_n : \alpha \text{ has } \leq r \text{ transverse blocks} \}
\]
for \( 0 \leq r \leq n \).

Theorem (E+Gray, 2014)
If \( 0 \leq r \leq n - 1 \), then
\[
\text{rank}(I_r) = \text{idrank}(I_r) = n \sum_{j=r}^{n} S(n, j) \cdot \binom{j}{r}
\]
The idempotent generating sets of this size have not been classified/enumerated (for \( 1 \leq r \leq n - 2 \)).
The ideals of $\mathcal{P}_n$ are

$$I_r = \{ \alpha \in \mathcal{P}_n : \alpha \text{ has } \leq r \text{ transverse blocks} \}$$

for $0 \leq r \leq n$. 

Theorem (E+Gray, 2014) If $0 \leq r \leq n - 1$, then $I_r$ is idempotent generated, and $\text{rank}(I_r) = \text{idrank}(I_r) = n \sum_{j=r}^{n} S(n, j)$.
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If $0 \leq r \leq n - 1$, then $I_r$ is idempotent generated, and

$$\text{rank}(I_r) = \text{idrank}(I_r) = \sum_{j=r}^{n} S(n, j) \binom{j}{r}.$$
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\]

for \( 0 \leq r \leq n \).

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\]

The idempotent generating sets of this size have not been classified/enumerated (for \( 1 \leq r \leq n - 2 \)).
Theorem (Maltcev and Mazorchuk, 2007)

$B_n \setminus S_n$ is idempotent generated.

$$B_n \setminus S_n = \langle u_{ij} | 1 \leq i < j \leq n \rangle.$$ 

$$\text{rank}(B_n \setminus S_n) = \text{idempotent rank}(B_n \setminus S_n) = \binom{n^2}{2} = n(n-1)/2.$$
Theorem (Maltcev and Mazorchuk, 2007)

- \( \mathcal{B}_n \setminus \mathcal{S}_n \) is idempotent generated.
- \( \mathcal{B}_n \setminus \mathcal{S}_n = \langle u_{ij} \mid 1 \leq i < j \leq n \rangle \).

\[ u_{ij} = \]

\[
\begin{array}{cccc}
1 & \cdots & i & \cdots & j & \cdots & n \\
& \| & \cdots & \| & \cdots & \| \\
& & \cdots & \cdots & \cdots & \cdots \\
& & & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]
Theorem (Maltcev and Mazorchuk, 2007)

- $\mathcal{B}_n \setminus \mathcal{S}_n$ is idempotent generated.
- $\mathcal{B}_n \setminus \mathcal{S}_n = \langle u_{ij} \ (1 \leq i < j \leq n) \rangle$.
- $\text{rank}(\mathcal{B}_n \setminus \mathcal{S}_n) = \text{idrank}(\mathcal{B}_n \setminus \mathcal{S}_n) = \binom{n}{2} = \frac{n(n-1)}{2}$. 

Diagram:

```
  1   i   j   n
<p>| | | |</p>
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  1   2   3
```

- $u_{ij} = \begin{array}{c}
  1 \\
  \vdots \\
  i \\
  \vdots \\
  j \\
  \vdots \\
  n$

James East

Idempotent generation in partition monoids
Let $\Lambda_n$ be the di-graph with vertex set

$$V(\Lambda_n) = \{A \subseteq n : |A| = 2\}$$

and edge set

$$E(\Lambda_n) = \{(A, B) : A \cap B \neq \emptyset\}.$$
Theorem (E+Gray, 2014)

The minimal idempotent generating sets of $\mathcal{B}_n \setminus \mathcal{S}_n$ are in one-one correspondence with the permutation subgraphs of $\Lambda_n$.

No formula is known for the number of minimal idempotent generating sets of $\mathcal{B}_n \setminus \mathcal{S}_n$ (yet). Very hard!

There are (way) more than $(n - 1)! \cdot (n - 2)! \cdot \cdots \cdot 3! \cdot 2! \cdot 1!$.

- Thanks to James Mitchell for $n = 5, 6$ (GAP).
The ideals of $\mathcal{B}_n$ are

$$I_r = \{ \alpha \in \mathcal{B}_n : \alpha \text{ has } \leq r \text{ transverse blocks} \}$$

for $0 \leq r = n - 2k \leq n$.

**Theorem (E+Gray, 2014)**

If $0 \leq r = n - 2k \leq n - 2$, then $I_r$ is idempotent generated and

$$\text{rank}(I_r) = \text{idrank}(I_r) = \binom{n}{2k}(2k - 1)!! = \frac{n!}{2^k k! r!}.$$
Theorem (Borisavljević, Došen, Petrić, 2002, etc)

- $\mathcal{TL}_n$ is idempotent generated.
- $\mathcal{TL}_n = \langle u_1, \ldots, u_{n-1} \rangle$.
- $\mathcal{TL}_n$ is idempotent generated.

$u_i = \begin{array}{cccc}
1 & & & i \\
\text{{\ldots\ldots}} & \bullet & \text{{\ldots\ldots}} & \text{{\ldots\ldots}} \\
\end{array}$

- $\text{rank}(\mathcal{TL}_n) = \text{idrank}(\mathcal{TL}_n) = n - 1$. 
Let $\Xi_n$ be the di-graph with vertex set

$$V(\Xi_n) = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\}$$

and edge set

$$E(\Xi_n) = \{(A, B) : A \cap B \neq \emptyset\}.$$
Theorem (E+Gray, 2014)

The minimal idempotent generating sets of $\mathcal{TL}_n$ are in one-one correspondence with the permutation subgraphs of $\Xi_n$.

The number of minimal idempotent generating sets of $\mathcal{TL}_n$ is $F_n$, the $n$th Fibonacci number.

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<td>13</td>
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</table>
The ideals of $\mathcal{T}\mathcal{L}_n$ are

$$I_r = \{\alpha \in \mathcal{T}\mathcal{L}_n : \alpha \text{ has } \leq r \text{ transverse blocks}\}$$

for $0 \leq r = n - 2k \leq n$.

**Theorem (E+Gray, 2014)**

If $0 \leq r = n - 2k \leq n - 2$, then $I_r$ is idempotent generated and

$$\text{rank}(I_r) = \text{idrank}(I_r) = \frac{r + 1}{n + 1} \binom{n + 1}{k}.$$
Values of $\text{rank}(I_r) = \text{idrank}(I_r)$:

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I have more problems but I should stop now...
I have more problems but I should stop now...

...unless I have a few minutes to spare...
Theorem $\mathcal{P}_X = \langle S_X, \alpha, \beta \rangle$ where

\[
\alpha = \prod_{X} |X| \quad \text{and} \quad \beta = \prod_{X} |X|.
\]
Infinite partition monoids — $\mathcal{P}_X$

**Theorem**

$\mathcal{P}_X = \langle \mathcal{S}_X, \alpha, \beta \rangle$ where

\[ \alpha = \]

\[ \beta = \]
Proof: Let $\gamma \in \mathcal{P}_X$. 
Proof: Let $\gamma \in \mathcal{P}_X$. 

\[
\begin{array}{c}
A \\
\hline
B
\end{array}
\quad \cdots \quad
\begin{array}{c}
C \\
D
\end{array}
\]
Proof: Let $\gamma \in \mathcal{P}_X$. We’ll show that $\gamma = \alpha\pi\beta$ for some $\pi \in S_X$. 
Proof: Let $\gamma \in \mathcal{P}_X$. We’ll show that $\gamma = \alpha \pi \beta$ for some $\pi \in S_X$. 

Diagram: 

\[ \gamma \] 

\[ \alpha \] 

\[ \beta \]
Proof: Let $\gamma \in \mathcal{P}_X$. We’ll show that $\gamma = \alpha \pi \beta$ for some $\pi \in S_X$. 

\begin{align*}
\gamma & \quad \begin{array}{c}
A \\
\hline
B \\
\end{array} \\
\alpha & \quad \begin{array}{c}
A \\
\hline
B \\
\end{array} \begin{array}{c}
C \\
\hline
D \\
\end{array} \\
\beta & \quad \begin{array}{c}
C \\
\hline
D \\
\end{array}
\end{align*}
Proof: Let $\gamma \in \mathcal{P}_X$. We’ll show that $\gamma = \alpha \pi \beta$ for some $\pi \in S_X$. 

\[ \begin{array}{c}
\gamma \\
A \\
B \\
C \\
D \\
a \\
b \\
B \\
C \\
D
\end{array} \]
Proof: Let $\gamma \in \mathcal{P}_X$. We’ll show that $\gamma = \alpha \pi \beta$ for some $\pi \in S_X$. 

\[ \begin{array}{c}
\gamma \\
\hline
A \\
\{B\} \\
\hline
\hline
C \\
\{D\}
\end{array} \]
Proof: Let $\gamma \in \mathcal{P}_X$. We’ll show that $\gamma = \alpha \pi \beta$ for some $\pi \in S_X$. 

\[ \gamma \]

\[ \alpha \]

\[ \beta \]
Proof: Let $\gamma \in \mathcal{P}_X$. We’ll show that $\gamma = \alpha \pi \beta$ for some $\pi \in S_X$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\end{figure}
Proof: Let $\gamma \in \mathcal{P}_X$. We’ll show that $\gamma = \alpha \pi \beta$ for some $\pi \in S_X$. 

\[
\begin{array}{c}
\gamma \\
\hline
A \quad B \\
\hline
C \quad D
\end{array}
\]

\[
\begin{array}{c}
\alpha \\
\hline
A \quad B \\
\hline
C \quad D
\end{array}
\]

\[
\begin{array}{c}
\beta \\
\hline
A \quad B \\
\hline
C \quad D
\end{array}
\]
Proof: Let $\gamma \in \mathcal{P}_X$. We’ll show that $\gamma = \alpha \pi \beta$ for some $\pi \in S_X$. 
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\[ \begin{array}{c}
\gamma \\
A \\
B \\
C \\
D \\
\end{array} \] 

\[ \begin{array}{c}
\alpha \\
A \\
B \\
\end{array} \] 

\[ \begin{array}{c}
\beta \\
C \\
D \\
\end{array} \]
Proof: Let $\gamma \in \mathcal{P}_X$. We’ll show that $\gamma = \alpha \pi \beta$ for some $\pi \in S_X$. 
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Proof: Let $\gamma \in \mathcal{P}_X$. We’ll show that $\gamma = \alpha \pi \beta$ for some $\pi \in S_X$. 
Let $\alpha \in \mathcal{P}_X$. 

Write $\alpha = (A_i \cap B_i, C_j \cap D_k)$ for $i \in I, j \in J, k \in K$. 

Define:

- $\text{def}(\alpha) = \sum_{j \in J} |C_j|$
- $\text{codef}(\alpha) = \sum_{k \in K} |D_k|$
- $\text{col}(\alpha) = \sum_{i \in I} (|A_i| - 1)$
- $\text{cocol}(\alpha) = \sum_{i \in I} (|B_i| - 1)$
- $\text{sh}(\alpha) = \# \{i \in I : A_i \cap B_i = \emptyset\}$
Let $\alpha \in \mathcal{P}_X$.

$\alpha = \alpha_{i,j,k}$

Define:
- $\text{def}(\alpha) = \sum_{j \in J} |C_j|$
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Infinite partition monoids — $\mathcal{P}_X$

Let $\alpha \in \mathcal{P}_X$.

\[ \alpha = \begin{array}{c}
\ldots \ldots \\
\quad A_i \quad C_j \\
\quad \ldots \ldots \\
\quad B_i \quad D_k \\
\end{array} \]

- Write $\alpha = \left( \begin{array}{c|c}
A_i & C_j \\
B_i & D_k \\
\end{array} \right)_{i \in I, \ j \in J, \ k \in K}. $
Let $\alpha \in \mathcal{P}_{\chi}$.

Write $\alpha = \left( \begin{array}{c|c} A_i & C_j \\ \hline B_i & D_k \end{array} \right)_{i \in I, j \in J, k \in K}$.

Define:

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Write $\alpha = \left( \begin{array}{c|c} A_i & C_j \\ \hline B_i & D_k \end{array} \right)_{i \in I, j \in J, k \in K}$.

Define:
- $\text{def}(\alpha) = \sum_{j \in J} |C_j|$, $\text{codef}(\alpha) = \sum_{k \in K} |D_k|$,
Let $\alpha \in \mathcal{P}_X$.

\[ \alpha = \begin{array}{c}
A_i \\
B_i \\
C_j \\
D_k
\end{array} \]

- Write $\alpha = \left( \begin{array}{c|c}
A_i & C_j \\
B_i & D_k
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  - $\text{def}(\alpha) = \sum_{j \in J} |C_j|$, $\text{cdef}(\alpha) = \sum_{k \in K} |D_k|$,
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Let $\alpha \in \mathcal{P}_X$.

Write $\alpha = \left( \begin{array}{c|c} A_i & C_j \\ \hline B_i & D_k \end{array} \right)_{i \in I, j \in J, k \in K}$.

Define:

- $\text{def}(\alpha) = \sum_{j \in J} |C_j|$,  $\text{codef}(\alpha) = \sum_{k \in K} |D_k|$, 
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Let $\alpha \in \mathcal{P}_X$.

Write $\alpha = \left( \begin{array}{c|c} A_i & C_j \\ \hline B_i & D_k \end{array} \right)_{i \in I, \ j \in J, \ k \in K}$.

Define:

- $\text{def}(\alpha) = \sum_{j \in J} |C_j|$, $\text{cdef}(\alpha) = \sum_{k \in K} |D_k|$,  
- $\text{col}(\alpha) = \sum_{i \in I} (|A_i| - 1)$, $\text{cocol}(\alpha) = \sum_{i \in I} (|B_i| - 1)$,  
- $\text{sh}(\alpha) = \# \{ i \in I : A_i \cap B_i = \emptyset \}$. 

James East

Idempotent generation in partition monoids
Theorem (E+FitzGerald, 2012)

If $X$ is infinite, then

$$\langle E(\mathcal{P}_X) \rangle = \{1\} \cup (\mathcal{P}^\text{fin}_X \setminus \mathcal{S}^\text{fin}_X) \cup \left\{ \alpha \in \mathcal{P}_X : \begin{array}{c} \text{col}(\alpha) + \text{def}(\alpha) = \text{cocol}(\alpha) + \text{codef}(\alpha) \\ \geq \max(\text{sh}(\alpha), \aleph_0) \end{array} \right\}.$$
### Theorem (E+FitzGerald, 2012)

If $X$ is infinite, then

$$
\langle E(\mathcal{P}_X) \rangle = \{1\} \cup (\mathcal{P}_X^{\text{fin}} \setminus \mathcal{S}_X^{\text{fin}}) \cup \left\{ \alpha \in \mathcal{P}_X : \text{col}(\alpha) + \text{def}(\alpha) = \text{cocol}(\alpha) + \text{codef}(\alpha) \geq \max(\text{sh}(\alpha), \aleph_0) \right\}.
$$

### Theorem (E+FitzGerald, 2012)

For any $X$ (finite or infinite),

$$
\langle S_X \cup E(\mathcal{P}_X) \rangle = \{ \alpha \in \mathcal{P}_X : \text{col}(\alpha) + \text{def}(\alpha) = \text{cocol}(\alpha) + \text{codef}(\alpha) \}.
$$
Thanks for having me in Stuttgart!