Cellularity of Wreath Product Algebras

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The main goal of this work is to study cellular structure of the wreath product algebras $A \wr S_n$. For this we introduce a variant of the notion of cellularity called cyclic cellularity: A cellular algebra $A$ is called cyclic cellular if all of its cell modules are cyclic $A$–modules.

Although it seems to be stronger than cellularity, it includes most of the important classes of cellular algebras appearing already in the literature. For example, Hecke algebras of type $A$, $q$–Schur algebras, Brauer algebras and BMW algebras are cyclic cellular.
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Definition

Let $A$ a unital $R$–algebra over an integral domain $R$. A cell datum for $A$ consists of an $R$–linear algebra involution $a \mapsto a^*$; a finite poset $(\Gamma, \geq)$; for each $\gamma \in \Gamma$ a finite index set $\mathcal{T}(\gamma)$; and a subset

$$\mathcal{C} = \{ c_{s,t}^\gamma : \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma) \} \subseteq A$$

with the following properties:

(1) $\mathcal{C}$ is an $R$–basis of $A$.

(2) For each $\gamma \in \Gamma$, let $\bar{A}^\gamma$ be the span of the $c_{s,t}^\mu$ with $\mu > \gamma$. 

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with the following properties:

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(defn. contd.) for $a \in A$

$$ac_{s,t}^\gamma \equiv \sum_v r_v^s(a)c_{v,t}^\gamma \mod \bar{A}^\gamma.$$ 

where the co-efficients in the expansion are independent of $t$.

$$(3) \quad (c_{s,t}^\gamma)^* \equiv c_{t,s}^\gamma \mod \bar{A}^\gamma \text{ for all } \gamma \in \Gamma \text{ and, } s, t \in T(\gamma).$$

The original definition of Graham and Lehrer includes a stronger version of condition (3), as follows:

$$(3') \quad (c_{s,t}^\gamma)^* = c_{t,s}^\gamma \text{ for all } \gamma \in \Gamma \text{ and } s, t \in T(\gamma).$$

For brevity, $(C, \Gamma)$ is a cellular basis of $A$ and $(A, *, \Gamma, \geq, T, C)$ is a cell datum for $A$. 
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(3') $(c_{s,t}^\gamma)^* = c_{t,s}^\gamma$ for all $\gamma \in \Gamma$ and $s, t \in T(\gamma).$

For brevity, $(C, \Gamma)$ is a cellular basis of $A$ and $(A, *, \Gamma, \succeq, T, C)$ is a cell datum for $A$. 
Given $\gamma \in \Gamma$, let $A\gamma$ cell ideal of $A$ denote the span of the $c^{\mu}_{s,t}$ with $\mu \geq \gamma$.

For $\gamma \in \Gamma$, the left cell module $\Delta\gamma$ is defined as follows:

1. as an $R$–module, $\Delta\gamma$ is free with basis indexed by $T(\gamma)$, say $\{c^{\gamma}_{s} : s \in T(\gamma)\}$;

2. for each $a \in A$, the action of $a$ on $\Delta\gamma$ is defined by $ac^{\gamma}_{s} = \sum_{\nu} r_{\nu}^{s}(a)c^{\gamma}_{\nu}$ where the elements $r_{\nu}^{s}(a) \in R$ are the coefficients in the definition.
Given $\gamma \in \Gamma$, let $A^\gamma$ cell ideal of $A$ denote the span of the $c_{s,t}^{\mu}$ with $\mu \geq \gamma$.

For $\gamma \in \Gamma$, the left cell module $\Delta^\gamma$ is defined as follows:

1. as an $R$–module, $\Delta^\gamma$ is free with basis indexed by $T(\gamma)$, say $\{c_s^\gamma : s \in T(\gamma)\}$;
2. for each $a \in A$, the action of $a$ on $\Delta^\gamma$ is defined by $ac_s^\gamma = \sum_v r_v^s(a) c_v^\gamma$ where the elements $r_v^s(a) \in R$ are the coefficients in the definition.
Let $A$ be a cellular algebra with cell datum $(\mathcal{A}, \ast, \Gamma, \geq, \mathcal{T}, \mathcal{C})$. We say that a cellular basis

$$\mathcal{B} = \{ b_{s, t}^\gamma : \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma) \}$$

is *equivalent* to the original cellular basis $\mathcal{C}$ if it determines the same ideals $A^\gamma$ and the same cell modules as does $\mathcal{C}$. More precisely, the requirement is that

1. for all $\gamma \in \Gamma$,

$$A^\gamma = \text{span}\{ b_{s, t}^{\gamma'} : \gamma' \geq \gamma \text{ and } s, t \in \mathcal{T}(\gamma') \},$$

and

2. for all $\gamma \in \Gamma$ and $t \in \mathcal{T}(\gamma)$,

$$\text{span}\{ b_{s, t}^\gamma + \bar{A}^\gamma : s \in \mathcal{T}(\gamma) \} \cong \Delta^\gamma,$$ as $A$–modules.
Let \( A \) be a cellular algebra with cell datum \((A, \ast, \Gamma, \geq, T, C)\). We say that a cellular basis

\[
\mathcal{B} = \{ b_{s,t}^\gamma : \gamma \in \Gamma \text{ and } s, t \in T(\gamma) \}
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is \textit{equivalent} to the original cellular basis \( C \) if it determines the same ideals \( A^\gamma \) and the same cell modules as does \( C \). More precisely, the requirement is that

1. for all \( \gamma \in \Gamma \),

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A^\gamma = \text{span}\{ b_{s,t}^\gamma' : \gamma' \geq \gamma \text{ and } s, t \in T(\gamma') \}, \text{ and}
\]

2. for all \( \gamma \in \Gamma \) and \( t \in T(\gamma) \),

\[
\text{span}\{ b_{s,t}^\gamma + \bar{A}^\gamma : s \in T(\gamma) \} \cong \Delta^\gamma, \text{ as } A\text{–modules}.
\]
A cellular algebra $A$ always admits many different cellular basis. In fact, any choice of an $R$–basis in each cell module can be globalized to a cellular basis of $A$ as follows.

**Lemma**

Let $A$ be a cellular algebra with cell datum $(A, \ast, \Gamma, \geq, \mathcal{T}, \mathcal{C})$. For each $\gamma \in \Gamma$, fix an $A$–$A$ bimodule isomorphism $\beta_\gamma : A^\gamma / \bar{A}^\gamma \to \Delta^\gamma \otimes_R (\Delta^\gamma)^*$ satisfying $\ast \circ \beta_\gamma = \beta_\gamma \circ \ast$, and let $\{b_t : t \in \mathcal{T}(\gamma)\}$ be an $R$–basis of $\Delta^\gamma$. Finally, for each $\gamma \in \Gamma$ and each pair $s, t \in \mathcal{T}(\gamma)$, let $b^\gamma_{s,t}$ be an arbitrary lifting of $\beta_\gamma^{-1}(b_s \otimes b_t^*)$ in $A^\gamma$. Then

$$\mathcal{B} = \{b^\gamma_{s,t} : \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma)\}$$

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A cellular algebra is said to be *cyclic cellular* if every cell module of $A$ is cyclic.

A cellular basis is cyclic cellular if the cell modules defined via this basis are cyclic.
Lemma

Let $A$ be a cellular algebra over an integral domain $R$ with cell datum $(A, *, \Gamma, \geq, T, C)$. The following are equivalent:

1. $A$ is cyclic cellular.
2. For each $\gamma \in \Gamma$, there exists an element $y_\gamma \in A^\gamma$ with the properties:
   - $y_\gamma \equiv y_\gamma^* \mod \tilde{A}^\gamma$.
   - $A^\gamma = Ay_\gamma A + \tilde{A}^\gamma$.
   - $(Ay_\gamma + \tilde{A}^\gamma)/\tilde{A}^\gamma \cong \Delta^\gamma$, as $A$–modules.
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   3. $(A y_\gamma + \bar{A}^\gamma)/\bar{A}^\gamma \cong \Delta^\gamma$, as $A$–modules.
For each $\gamma \in \Gamma$, let $\delta^\gamma$ be a generator of the cell module $\Delta^\gamma$, and let $y_\gamma$ be a lifting in $A^\gamma$ of $\alpha^{-1}_\gamma(\delta^\gamma \otimes (\delta^\gamma)^*)$.

Let $\{c^\gamma_t : t \in T(\gamma)\}$ be the standard basis of the cell module $\Delta^\gamma$ derived from the cellular basis $C$ of $A$. Since $\Delta^\gamma$ is cyclic, there exist elements $v_t \in A$ such that $c^\gamma_t = v_t \delta^\gamma$. We denote

$$V^\gamma = \{v_t : t \in T(\gamma)\}.$$
For each $\gamma \in \Gamma$, let $\delta^\gamma$ be a generator of the cell module $\Delta^\gamma$, and let $y_\gamma$ be a lifting in $A^\gamma$ of $\alpha_\gamma^{-1}(\delta^\gamma \otimes (\delta^\gamma)^*)$.

Let $\{c_t^\gamma : t \in T(\gamma)\}$ be the standard basis of the cell module $\Delta^\gamma$ derived from the cellular basis $C$ of $A$. Since $\Delta^\gamma$ is cyclic, there exist elements $\nu_t \in A$ such that $c_t^\gamma = \nu_t \delta^\gamma$. We denote

$$V^\gamma = \{\nu_t : t \in T(\gamma)\}.$$
Lemma

For each $\gamma \in \Gamma$, let $\{b_t : t \in \mathcal{T}(\gamma)\}$ be an $R$–basis of the cell module $\Delta^\gamma$. For $t \in \mathcal{T}(\gamma)$, choose $v'_t \in A$ such that $b_t = v'_t \delta^\gamma$.

For $s, t \in \mathcal{T}(\gamma)$, let $b^\gamma_{s,t} = v'_s y^\gamma(v'_t)^*$. Then $\mathcal{B} = \{b^\gamma_{s,t} : \gamma \in \Gamma \text{ and } s, t \in \mathcal{T}(\gamma)\}$ is a cellular basis of $A$ equivalent to the original cellular basis $\mathcal{C}$. 
\( R\mathfrak{S}_n \) with Murphy basis is an example of a cyclic cellular algebra.

1. \( \Lambda \) is set of partitions of \( n \) with dominance order, \( T(\lambda) \) is the set of standard \( \lambda \)-tableaux and \( * \) is such that \( \pi^* = \pi^{-1} \) for \( \pi \in \mathfrak{S}_n \).

2. For \( \lambda \in \Lambda \), \( x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} w \), where \( \mathfrak{S}_\lambda \) is the row stabilizer of \( t^\lambda \). Let \( d(t) \) be the unique permutation such that \( t = d(t)t^\lambda \).

3. For \( s, t \in T(\lambda) \), define

\[
m^\lambda_{s, t} = d(s)x_\lambda d(t)^*.
\]

4. The cell module \( \Delta^\lambda \) is spanned by \( \{d(s)x_\lambda + \overline{R\mathfrak{S}_n}^\lambda : s \in T(\lambda)\} \). The cell module \( \Delta^\lambda \) is evidently cyclic with generator \( x_\lambda + \overline{R\mathfrak{S}_n}^\lambda \).
$\mathbb{R}\mathcal{S}_n$ with Murphy basis is an example of a cyclic cellular algebra.

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3. For $s, t \in T(\lambda)$, define

$$m^\lambda_{s,t} = d(s)x_\lambda d(t)^*.$$

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$$\{d(s)x_\lambda + R\mathfrak{S}_n^\lambda : s \in T(\lambda)\}.$$ The cell module $\Delta^\lambda$ is evidently cyclic with generator $x_\lambda + R\mathfrak{S}_n^\lambda$. 

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Cellularity of Wreath Product Algebras
Let $A$ be an $R$–algebra. The wreath product is $A \wr S_n = A^\otimes n \rtimes S_n$ where $S_n$ acts on $A^\otimes n$ by place permutations. If $A$ is an algebra with involution $\ast$, then $S_n$ acts by $\ast$–preserving automorphisms and the wreath product is also an algebra with involution determined by

$((a_1 \otimes \cdots \otimes a_n)\pi)\ast = \pi^{-1}(a_1^* \otimes \cdots \otimes a_n^*)$

$= \pi^{-1}(a_1^* \otimes \cdots \otimes a_n^*)\pi^{-1}$.

**Theorem**

Let $A$ be a cyclic cellular algebra. Then for all $n \geq 1$, the wreath product algebra $A \wr S_n$ is a cyclic cellular algebra.
Let $\Lambda_n^\Gamma$ denote the set of maps $\lambda$ from $\Gamma$ to the set of partitions such that $\sum_{\gamma \in \Gamma} |\lambda(\gamma)| = n$.

By fixing a listing of $\Gamma$ consistent with it’s partial order, in the sense that $\gamma(i) \geq \gamma(j) \implies i \leq j$.

, we can identify $\Lambda_n^\Gamma$ with multi-partitions via

$$\lambda^{(i)} = \lambda(\gamma(i)) \quad (1 \leq i \leq r).$$

Then for each $\lambda \in \Lambda_n^\Gamma$, we have analogues of elements $x_\lambda, s, t^\lambda$ and $d(\bar{s})$ as mentioned earlier.
We generalize the dominance order $\succeq$ on multi-partitions, called $\Gamma$-dominance order $\succeq_{\Gamma}$: if for all $\gamma \in \Gamma$, and for all $j \geq 0$,

\[
\sum_{\gamma' > \gamma} |\lambda(\gamma')| + \sum_{i \leq j} \lambda(\gamma)_i \geq \sum_{\gamma' > \gamma} |\mu(\gamma')| + \sum_{i \leq j} \mu(\gamma)_i.
\]

which takes into account of the partial order on $\Gamma$ also.

Let $A$ be cyclic cellular with basis ($C$, $\Gamma$) with $r$ elements in it’s poset.

For $\lambda = (\lambda(\gamma_1), \cdots, \lambda(\gamma_r))$ let $\alpha(\lambda) = (|\lambda(\gamma_1)|, \cdots, |\lambda(\gamma_r)|)$. 
Let \( V^\alpha \) be the set of simple tensors in \( A \otimes^n \) whose first \( \alpha_1 \) tensorands belong to \( V^{\gamma(1)} \), the next \( \alpha_2 \) tensorands belong to \( V^{\gamma(2)} \), and so forth. Set

\[
y^\alpha = y^{\otimes \alpha_1}_{\gamma(1)} \otimes \cdots \otimes y^{\otimes \alpha_r}_{\gamma(r)}.
\]

For \( \lambda \in \Lambda^\Gamma_n \), let \( \mathcal{T}(\lambda) \) denote the set of pairs \((s, v)\), where \( s \) is a standard \( \lambda \)-tableau and \( v \in V^\alpha(\lambda) \).

Define

\[
m^\lambda_{(s,v),(t,w)} = d(s) v y^\alpha \chi_{\lambda} w^* d(t)^*,
\]

where \( \lambda \in \Lambda^\Gamma_n \), \( \alpha = \alpha(\lambda) \), \( s, t \) are row standard \( \lambda \)-tableaux, and \( v, w \in V^\alpha \).
Cell modules of $A \wr S_n$

Let $E_1, \ldots, E_r$ be a collection of $A$–modules. For a multipartition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of total size $n$ with $r$ parts, $\alpha = \alpha(\lambda)$. Then

$$\Delta^\lambda_R = \Delta^\lambda_R^{(1)} \otimes \cdots \otimes \Delta^\lambda_R^{(r)},$$

is a cell module for $R \wr S_\alpha \cong R \wr S_{\alpha_1} \otimes \cdots \otimes R \wr S_{\alpha_r}$. Let $E^\alpha = E_1^{\otimes \alpha_1} \otimes \cdots \otimes E_r^{\otimes \alpha_r}$. Then $E^\alpha$ is an $A \wr S_\alpha$–module, with $A^{\otimes n}$ acting by the tensor product action and $S_\alpha$ acting by place permutations. Moreover, $E^\alpha \otimes \Delta^\lambda_R$ is also an $A \wr S_\alpha$–module, with $a(v \otimes m) = av \otimes m$ and $\pi(v \otimes m) = \pi v \otimes \pi m$, for $a \in A^{\otimes n}$, $\pi \in S_\alpha$, $v \in E^\alpha$ and $m \in \Delta^\lambda_R$. 

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We obtain an $A \wr \mathfrak{S}_n$–module by

$$\text{Ind}_{A \wr \mathfrak{S}_n}^{A \wr \mathfrak{S}_n} (E^\alpha \otimes \Delta^\lambda_R) = (A \wr \mathfrak{S}_n) \otimes_{A \wr \mathfrak{S}_n} (E^\alpha \otimes \Delta^\lambda_R).$$

When this construction is applied to the simple modules of $A$, one obtains the simple modules of the wreath product $A \wr \mathfrak{S}_n$.

**Theorem**

Let $\lambda \in \Lambda^\Gamma_n$ and let $\alpha = \alpha(\lambda)$. The cell module $C^\lambda$ of $A \wr \mathfrak{S}_n$ satisfies

$$C^\lambda \cong \text{Ind}_{A \wr \mathfrak{S}_n}^{A \wr \mathfrak{S}_n} (E^\alpha \otimes \Delta^\lambda_R).$$