On the $U$-module Structure of the Unipotent Specht Modules for Finite General Linear Groups

Qiong Guo
(joint work with Richard Dipper)

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Basic setting

- $q$: a fixed power of some prime $p$.
- $\mathbb{F}_q$: finite field with $q$ elements.
- $K$: a field such that $\text{char}(K) \neq p$ and $\sqrt{1} \in K$.
- $G = \text{GL}_n(q)$: group of invertible $n \times n$ matrices over $\mathbb{F}_q$, where $n \in \mathbb{N}$. 

For $\lambda \vdash n$ (partition of $n$):
- Let $B \subseteq \mathcal{P}_\lambda$ be the corresponding parabolic subgroup,
- $K \mathcal{P}_\lambda$ the trivial $K \mathcal{P}_\lambda$-module.
- Let $M_{K}(\lambda) = \text{Ind}_{G \mathcal{P}_\lambda}^{G}(K \mathcal{P}_\lambda)$, the corresponding permutation module.
- The unipotent Specht module $S_{K}(\lambda)$ is a submodule of $M_{K}(\lambda)$ and for $K = \mathbb{C}$, $\{S_{\mathbb{C}}(\lambda) \mid \lambda \vdash n\}$ are precisely the irreducible constituents of $\text{Ind}_{G B}^{G}(C B)$. 

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If we set $q \rightarrow 1$, $S_K(\lambda) = S^\lambda$, the Specht module for the symmetric group $\mathfrak{S}_n$. 

Conjecture (Dipper-James, 1990)

There is a $q$-analogue of the standard basis theorem for $G$.

Those come with a natural integrally defined basis, called “standard basis”.

Theorem (Brandt-Dipper-James-Lyle, 2006)

DJ's conjecture holds for 2-part partitions.

Difficulty: The proof of this theorem is completely combinatorial and seems not work for general $\lambda$.

Goal

Reprove DJ's conjecture for 2-part partitions by using a method tightly connected to representation theory.
If we set $q \rightsquigarrow 1$, $S_K(\lambda) = S^\lambda$, the Specht module for the symmetric group $\mathfrak{S}_n$.

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Motivation

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Reprove DJ’s conjecture for 2-part partitions by using a method tightly connected to representation theory.
Normal form of an \((m \times n)\)-matrix

Let \(\lambda = (n - m, m) \vdash n\). Choose an \(m\)-dimensional \(\mathbb{F}_q\)-vector space \(V_1\) in \(V = \mathbb{F}_q^n\). List a basis of \(V_1\) as \(m \times n\)-matrix and then row reduce it to a unique normal form.

\[
\begin{bmatrix}
\ast & \ast & 1 & 0 & 0 & 0 & 0 \\
\ast & \ast & 0 & 1 & 0 & 0 & 0 \\
\ast & \ast & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

Example

We label the rows by column indices of “last 1’s”. Write \(\text{tab}(L) = 1 2 4 7 3 5 6 \in R_{\text{Std}}(\lambda)\) where \(\lambda = (4, 3)\).
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**Example**

\[
L = \begin{pmatrix}
  1 & 2 & 3 & 4 & 5 & 6 & 7 \\
  * & * & 1 & 0 & 0 & 0 & 0 \\
  * & * & 0 & * & 1 & 0 & 0 \\
  * & * & 0 & * & 0 & 1 & 0
\end{pmatrix}.
\]

We label the rows by column indices of “last 1’s”. Write

\[
\text{tab}(L) = \begin{pmatrix}
  1 & 2 & 4 & 7 \\
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\end{pmatrix} \in \mathbb{RStd}(\lambda) \text{ where } \lambda = (4, 3).
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Different description of a basis of $M_K(\lambda)$

**Definition**

$$\mathcal{X}_{m,n} = \{ \text{row reduced } m \times n - \text{matrices} \}$$

$$\mathcal{F}(\lambda) = \{ 0 \subseteq V_1 \subseteq V = \mathbb{F}_q^n | \dim_{\mathbb{F}_q} V_1 = m \}$$
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$G$ acts on $\mathcal{X}_{m,n}$ by setting $L \circ g$ for $L \in \mathcal{X}_{m,n}, g \in G$ to be the row reduced matrix obtained from $Lg$. The resulting $G$-permutation module is exactly $M_K(\lambda) = \text{Ind}_P^G K_{P\lambda}$. 

Remark: $\mathcal{X}_{m,n}$ is a basis of $M_K(\lambda)$. For $\lambda \vdash n$ arbitrary there is a similar description of a basis of $M_K(\lambda)$ by row reduced matrices.
Different description of a basis of $M_K(\lambda)$

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Definition

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\( G \) acts on \( \mathcal{X}_{m,n} \) by setting \( L \circ g \) for \( L \in \mathcal{X}_{m,n}, g \in G \) to be the row reduced matrix obtained from \( Lg \). The resulting \( G \)-permutation module is exactly \( M_K(\lambda) = \text{Ind}_{P_\lambda}^G K_{P_\lambda} \).

Remark

- \( \mathcal{X}_{m,n} \) is a basis of \( M_K(\lambda) \).
- For \( \lambda \vdash n \) arbitrary there is a similar description of a basis of \( M_K(\lambda) \) by row reduced matrices.
Define $\Phi_m : M_K(\lambda) \rightarrow M_K(\mu) : U \mapsto \sum_{X \subseteq U, \dim X = m-1} X$, where

$\dim U = m$ and $\mu = (n - m + 1, m - 1)$.
Define $\Phi_m : M_K(\lambda) \to M_K(\mu) : U \mapsto \sum_{X \subseteq U} X$, where $\dim U = m$ and $\mu = (n - m + 1, m - 1)$.

**Lemma**

For $K = \mathbb{C}$, we have $S_{\mathbb{C}}(\lambda) = \ker \Phi_m$. 
Define $\Phi_m : M_K(\lambda) \rightarrow M_K(\mu) : U \mapsto \sum_{X \subseteq U} X$, where $\dim U = m$ and $\mu = (n - m + 1, m - 1)$.

Lemma

For $K = \mathbb{C}$, we have $S_C(\lambda) = \ker \Phi_m$.

Strategy:

- **Step 1: Inspect** $\text{Res}_U^G M_K(\lambda)$. 
Define $\Phi_m : M_K(\lambda) \rightarrow M_K(\mu) : U \mapsto \sum_{X \subseteq U} X$, where $\dim U = m$ and $\mu = (n - m + 1, m - 1)$.

**Lemma**

For $K = \mathbb{C}$, we have $S_\mathbb{C}(\lambda) = \ker \Phi_m$.

**Strategy:**

- **Step 1:** Inspect $\text{Res}^G_U M_K(\lambda)$.
- **Step 2:** Using $\Phi_m$ to investigate $\text{Res}^G_U S_K(\lambda)$.
Define $\Phi_m : M_K(\lambda) \to M_K(\mu) : U \mapsto \sum_{X \subseteq U} X$, where
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**Strategy:**

- **Step 1:** Inspect $\text{Res}_U^G M_K(\lambda)$.
- **Step 2:** Using $\Phi_m$ to investigate $\text{Res}_U^G S_K(\lambda)$.

**Advantage:**

$|U| = p$-power, $\text{char}(K) \neq p \Rightarrow KU$ is semisimple.
Note that each row reduced $m \times n$-matrix determines an unique row standard $\lambda$-tableau.

**Definition**

For $t \in \text{RStd}(\lambda)$, denote the set of all the row reduced matrices which determine the same row standard $\lambda$-tableau $t$ by $X_t = \{ L \in X_{m,n} | \text{tab}(L) = t \}$ and set $M_t = KX_t$. 

$t$-batch $M_t$ of $MK(\lambda)$
Note that each row reduced $m \times n$-matrix determines an unique row standard $\lambda$-tableau.

**Definition**

For $t \in \mathbb{R}_{\text{Std}}(\lambda)$, denote the set of all the row reduced matrices which determine the same row standard $\lambda$-tableau $t$ by

$$\mathcal{X}_t = \{L \in \mathcal{X}_{m,n} | \text{tab}(L) = t\}$$

and set $\mathcal{M}_t = K \mathcal{X}_t$.

The $\mathcal{M}_t$ comes up naturally:

**Lemma**

$$\text{Res}_U^G \, M_K(\lambda) \cong \bigoplus_{w \in D_\lambda} \text{Ind}_{p_{\lambda \cap U}}^U K \stackrel{t=t^\lambda w}{\longrightarrow} \bigoplus_{t \in \mathbb{R}_{\text{Std}}(\lambda)} \mathcal{M}_t.$$  

$\mathcal{M}_t$ is called the $t$-batch of $M_K(\lambda)$. 

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Make $\mathcal{X}_t$ into an abelian $p$-group $(\mathcal{X}_t, \Diamond)$ by defining $L_1 \Diamond L_2$ as adding $L_1$ and $L_2$ in all columns except those, which contain a last one, keeping those unchanged. $K(\mathcal{X}_t, \Diamond)$ is commutative and semisimple, since $\text{char}(K) \neq p$ and $|\mathcal{X}_t|=$power of $q$. So:
Idempotent basis of $M_t$

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$M_t = KX_t$ has a $K$-basis $E_t$ consisting of primitive orthogonal idempotents $e_L$ where $L \in X_t$. 

**Advantage:** This new idempotent basis $E_t$ is more adaptable to the $U$-module structure.
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**Advantage:**

*This new idempotent basis \( \mathcal{E}_t \) is more adaptable to the \( U \)-module structure.*
Monomial action of $U^w \cap U$ on $E_t$

**Proposition**

Let $t = t^\lambda w \in RStd(\lambda)$. Then $U^w \cap U$ acts monomially on the idempotent basis $E_t = \{ e_L \mid L \in \mathcal{X}_t \}$.
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**Theorem**

Let $O$ be an $U^w \cap U$-orbit of $E_t$. Then the orbit module $M_O = K O$ is an irreducible $(U^w \cap U)$-module. Moreover $M_O$ is $U$-invariant, hence it is an irreducible $U$-module.
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**Remark**

We can classify these orbits and describe their sizes by “combinatorial data”.
Using those “combinatorial data”, we can prove the following conjecture for 2-part partitions:

**Conjecture (Dipper, G. 2011)**

Let $\lambda \vdash n$. Then there exists for each $0 \leq c \in \mathbb{Z}$ a polynomial $\ell_{c,\lambda}(t) \in \mathbb{Z}[t]$ depending only on $\lambda$, not on $q$ such that $\ell_{c,\lambda}(q)$ is the number of irreducible direct summands of $\text{Res}_U^G(S_K(\lambda))$ of dimension $q^c$. 
In order to prove this conjecture for 2-part partitions we use $\Phi_m$ to carry over from the permutation module $M_K(\lambda)$ to $S_K(\lambda)$. This works since the following holds:
Main result for $\lambda = (n - m, m)$

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**Proposition**

$\Phi_m$ preserves “combinatorial data”.

Using this and a theorem of Lyle we obtain in addition:

**Theorem**

For $\lambda = (n - m, m) \vdash n$, we construct an integral standard basis for $S_K(\lambda)$, reproving DJ's conjecture for 2-part partitions.

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This method works for arbitrary partition $\lambda \vdash n$. Of course more tools are needed in general.
Applied to $\lambda = (1^n)$, $s = t^\lambda$, the unique standard $\lambda$-tableau. Similarly we can prove that $U_n$ acts monomially on the idempotent basis corresponding to this.

**Example: Idempotent in the $s$-batch of $U_n$:**

$$e_L = \begin{array}{ccc}
1 & 2 & n \\
\lambda_1 & 1 & \\
\lambda_2 & \lambda_3 & 1 \\
\vdots & \vdots & \ddots \\
\lambda_n & \lambda_{n-1} & \cdots \\
\end{array} \in \mathcal{E}_s.$$
Supercharacter

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e_L = \begin{array}{cccc}
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Unfortunately the $U_n$-orbit modules are in this case not always irreducible, and their characters are called supercharacters. (André, Yan, Isaacs, Diaconis...)
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Unfortunately the $U_n$-orbit modules are in this case not always irreducible, and their characters are called supercharacters. (André, Yan, Isaacs, Diaconis...)

**Facts:** Supercharacters are also classified by “combinatorial datas” similarly.
Note that the unipotent Specht module $S_K(1^n)$ is the Steinberg module and its restriction to $U_n$ is the regular representation of $U_n$. Using number theory we obtain: Our conjecture contains as special case the following longstanding conjectures:

**Conjecture (Lehrer 1974)**
The number of distinct irreducible complex characters of degree $q^c$ of $U_n$ is a polynomial in $q$ with integral coefficients depending only on $n$ not on $q$.

**Conjecture (Higman 1960)**
The number of conjugacy classes of $U_n$ is a polynomial in $q$ with integral coefficients depending only on $n$ not on $q$. 
Related conjectures

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The number of conjugacy classes of $U_n$ is a polynomial in $q$ with integral coefficients depending only on $n$ not on $q$. 
Obviously, the problem of decomposing the orbit modules of the action of $U$ on the idempotent basis of $\text{Res}_U^G(S_K(1^n))$ turns into the problem of decomposing so called “supercharacters” of $U$ into irreducibles.
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**Proposition (Dipper,G. 2012)**

For $\lambda = (n - m, m) \vdash n$, $t \in \mathrm{RStd}(\lambda)$, $\mu = (1^n)$, $s = t^\mu$. Let $\mathcal{O} \subseteq \mathcal{E}_t$, $\tilde{\mathcal{O}} \subseteq \mathcal{E}_s$ be some orbits in $M_K(\lambda)$ and $M_K(\mu)$ respectively. Then the irreducible $\mathbb{C}U$-module $\mathbb{C}\mathcal{O}$ occurs as a constituent of the orbit module $\mathbb{C}\tilde{\mathcal{O}}$ if these two orbits have the same “combinatorial data”.

Not a surprise, but difficult to prove! This splits off one specific irreducible constituent from a supercharacter of $U$. 
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Not a surprise, but difficult to prove! This splits off one specific irreducible constituent from a supercharacter of $U$. 
Examples of “combinatorial data” for 2-part partitions

The orbits and their sizes (by example):

Let $\lambda = (2, 2)$, $t = \frac{1}{2} \frac{3}{4}$.

The irreducible orbit module modules occurring in the batch $\mathcal{M}_t$ can be classified by the following idempotents:

\[
\begin{array}{ccc}
  z & 1 & 1 \\
  1 & z & 1 \\
\end{array}
\quad
\begin{array}{ccc}
  1 & 1 & 1 \\
  y & z & 1 \\
\end{array}
\quad
\begin{array}{ccc}
  y & 1 & 1 \\
  z & z & 1 \\
\end{array}
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where $0 \neq z, y \in \mathbb{F}_q$. 
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\begin{array}{|c|c|}
\hline
z & 1 \\
\hline
1 & 1 \\
\hline
\end{array} \quad
\begin{array}{|c|c|}
\hline
1 & z \\
\hline
z & 1 \\
\hline
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\begin{array}{|c|c|}
\hline
y & 1 \\
\hline
z & 1 \\
\hline
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\begin{array}{|c|c|}
\hline
* & 1 \\
\hline
z & * \\
\hline
\end{array}
\]

where \( 0 \neq z, y \in \mathbb{F}_q \).

Monomial action = putting arbitrary values of \( \mathbb{F}_q \) to \( * \)-places. Hence we obtain the dimension of the corresponding orbit respectively by:

\[
1 \quad 1 \quad 1 \quad q^2
\]
Example

$$\lambda = (4, 3), s = \frac{1}{3} \frac{2}{5} \frac{4}{6} \frac{7}{\in RStd(\lambda), 0 \neq \alpha, \beta, \gamma \in \mathbb{F}_q}$$

$\mathbf{1} \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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$$e_L$$

$\mathbf{1} \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$

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$$e_{\hat{L}}$$

Then $e_L KU \leq e_{\hat{L}} KU$ (actually occurring with multiplicity 1).