Tilting Modules and Sheaves on Moment Graphs

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Gradings and Decomposition Numbers
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1. The category $\mathcal{O}_A$
2. Sheaves on moment graphs
3. Tilting modules as sheaves
4. Applications

Basic setting

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- A a local, commutative S-algebra with structure map $\tau : S \to A$
  (in this talk $A = S_{(0)}$ or $A = S_{(0)}/S_{(0)}\mathfrak{h} = \mathbb{C}$)

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- $\mathcal{O}_A \cong \bigoplus_\lambda \mathcal{O}_{A,\lambda}$ where λ runs over anti-dominant weights (block decomposition).

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- Define the order on $\mathcal{V}$ by

$$w \leq w' \Leftrightarrow w \cdot \lambda \leq w' \cdot \lambda$$

  for all $w, w' \in \mathcal{W}$.

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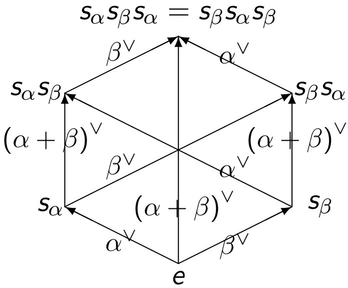

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For $A = S$ consider all S-modules and S-linear maps as graded modules and graded morphisms in degree zero.

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Denote by $\mathcal{B}^{\downarrow}(w)$ the Braden-MacPherson sheaf on the same moment graph $\mathcal{G}$ with reversed order.

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- $\mathbb{V}(\Delta_A(x \cdot \lambda)) = \mathcal{V}_A(x)$ *(sky scraper sheaf)*

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$$K_A(x \cdot \lambda) \otimes_A S_\alpha \cong \bigoplus_{w \in \mathcal{W}'} \left(K_{S_\alpha}(w \cdot \lambda)^{m_w} \oplus K_{S_\alpha}(s_\alpha w \cdot \lambda)^{n_w}\right)$$

where $\mathcal{W}' := \{w \in \mathcal{W} | w^{-1}\alpha \in R^+\}$ and $m_w, n_w \in \mathbb{N}$.

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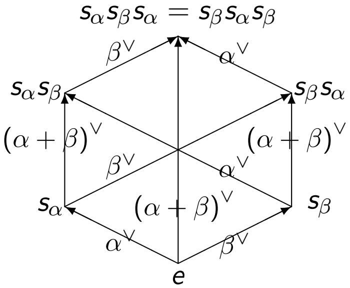

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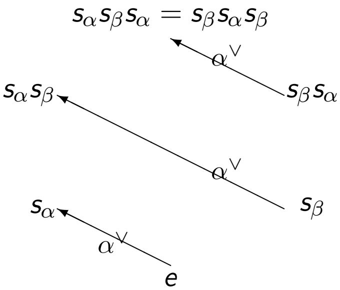
An $S_{(0)}$-sheaf $\mathcal{M}$ on becomes an S_α-sheaf on



after applying $\cdot \otimes_{S_{(0)}} S_\alpha$.

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Corollary

$$(K(x \cdot \lambda) : \Delta(y \cdot \lambda)) = P_{y,x}(1)$$

Andersen filtration

On the space $\mathrm{Hom}_{\mathcal{O}}(\Delta(y \cdot \lambda), K(x \cdot \lambda))$ Soergel defines the *Andersen filtration*

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Theorem (Soergel)

Denote by $\bar{F}^i \, \mathrm{Hom}_{\mathcal{O}}(\Delta(y \cdot \lambda), K(x \cdot \lambda))$ *the subquotients of the Andersen filtration.*

$$\sum_i \dim_{\mathbb{C}} \bar{F}^i \, \mathrm{Hom}_{\mathcal{O}}(\Delta(y \cdot \lambda), K(x \cdot \lambda)) q^{(l(x)-l(y)-i)/2} = P_{y,x}(q)$$

Andersen and Jantzen filtration

On the level of sheaves on moment graphs we can identify $\mathrm{Hom}_{\mathcal{O}}(\Delta(y \cdot \lambda), K(x \cdot \lambda))$ with the *costalk* $\mathcal{B}^{\downarrow}(x)^{y} \otimes_{S} \mathbb{C}$, where

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Andersen and Jantzen filtration

Theorem

There is an isomorphism

$$\varphi : \mathcal{B}^{\uparrow}(xw_{\circ})^{yw_{\circ}} \otimes_S \mathbb{C} \xrightarrow{\sim} \mathcal{B}^{\downarrow}(x)^{y} \otimes_S \mathbb{C}$$

of graded $\mathbb{C}$-vector spaces which interchanges the Jantzen with the Andersen filtration.

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