Cyclotomic quiver Hecke algebras of type A

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June 2012

Cyclotomic Hecke algebras of type $G(\ell, 1, n)$

As we did in the first talk, fix a ring $\mathcal{O}$ an invertible parameter $\xi \in R^\times$ and a multicharge $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_\ell) \in \mathbb{Z}^\ell$. For $k \in \mathbb{Z}$ define

$$[k] = \begin{cases} 1 + \xi + \cdots + \xi^{k-1}, & \text{if } k \geq 0, \\ -(\xi^k + \xi^{k+1} + \cdots + \xi^{-1}), & \text{if } k < 0. \end{cases}$$

The integral cyclotomic Hecke algebra $\mathcal{H}_n^\Lambda = \mathcal{H}_n^\Lambda(\xi; \boldsymbol{\kappa})$ is the unital associative algebra generated by $T_1, \dots, T_{n-1}, L_1, \dots, L_n$ with relations

$$\begin{aligned} &\textstyle\prod_{l=1}^{\ell}(L_1 - [\kappa_l]) = 0, && (T_r + 1)(T_r - \xi) = 0, \\ &L_1 T_1 L_1 T_1 = T_1 L_1 T_1 L_1, && T_s T_{s+1} T_s = T_{s+1} T_s T_{s+1} \\ &L_r L_t = L_t L_r, && T_r L_r + 1 = L_{r+1} T_r - (\xi - 1) L_{r+1}, \\ &T_r L_t = L_t T_r, && \text{if } t \neq r, r+1, \\ &T_r T_s = T_s T_r, && \text{if } |r - s| > 1. \end{aligned}$$

Let $e = \min\{\, k \geq 0 : [k] = 0 \,\}$, $I = \mathbb{Z}/e\mathbb{Z}$ and let $\Gamma = \Gamma_e$ be the oriented quiver with vertex set I and edges $i \to i + 1$. Up to isomorphism, $\mathcal{H}_n^\Lambda$ is determined by the dominant weight $\Lambda = \Lambda(\boldsymbol{\kappa}) = \sum_{l=1}^{\ell} \Lambda_{\kappa_l}$ for Γ_e.

To make the formulas nicer, we assume that $e > 0$ for this talk.

Brundan-Kleshchev's graded isomorphism theorem

Let $\mathcal{R}_n^\Lambda$ be the cyclotomic quiver Hecke algebra of type Γ.

Theorem (Brundan and Kleshchev)

Suppose that K is a field. Then $\mathcal{H}_n^\Lambda \cong \mathcal{R}_n^\Lambda$ is a $\mathbb{Z}$-graded algebra.

Brundan and Kleshchev prove this by giving explicit isomorphisms $\mathcal{R}_n^\Lambda \xrightarrow{\simeq} \mathcal{H}_n^\Lambda$ and $\mathcal{H}_n^\Lambda \xrightarrow{\simeq} \mathcal{R}_n^\Lambda$ — and some inspired brute force calculations.

One of the aims of todays talk is to use the classical representation theory of $\mathcal{H}_n^\Lambda$ to explain this isomorphism, and then reap some of the consequences of this new perspective.

Theorem (Brundan, Kleshchev, Wang)

There is a graded lift S^λ of the (ungraded) Specht module $\underline{S}^\lambda$.

The graded Specht module S^λ comes with a homogeneous basis $\{\, v_{\mathfrak{t}} : \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \,\}$, where $\deg v_{\mathfrak{t}} = \deg \mathfrak{t}$ and where $\deg \mathfrak{t}$ is defined combinatorially following Lascoux-Leclerc-Thibon/Misra-Miwa.

Gram determinants of Specht modules

The Specht module $\underline{S}^\lambda$ comes equipped with a bilinear form $\langle\ ,\ \rangle$ and

$$\operatorname{rad} \underline{S}^\lambda = \{\, x \in \underline{S}^\lambda : \langle x, y \rangle = 0 \text{ for all } y \in \underline{S}^\lambda \,\}$$

Let $\underline{G}^\lambda = \big(\langle m_{\mathfrak{s}}, m_{\mathfrak{t}} \rangle\big)_{\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})}$ be the Gram matrix of $\langle\ ,\ \rangle$

$\Longrightarrow \quad \dim \underline{D}^\mu = \operatorname{rank} \underline{G}^\mu$ since $\underline{D}^\mu = \underline{S}^\mu / \operatorname{rad} \underline{S}^\mu$

Let $\mathcal{O} = \mathbb{Z}[t, t^{-1}]$ and assume that $\kappa_{l+1} - \kappa_l \geq n$. Let $\mathcal{K} = \mathbb{Q}(t)$.

Theorem (James-Murphy, Dipper-James, James-M.)

Suppose that $\mathcal{H}_n^\Lambda = \mathcal{H}_n^\Lambda(\mathcal{K})$ and that $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. Then

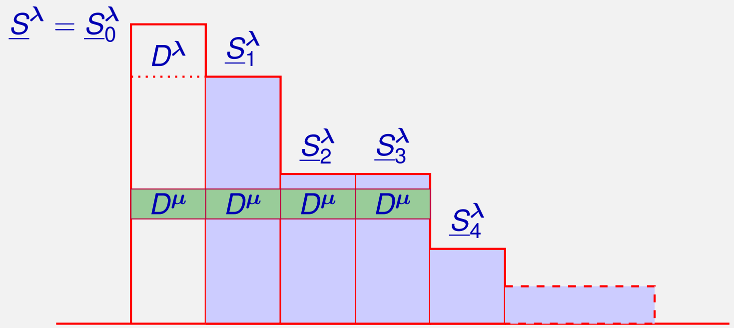
$$\det \underline{G}^\lambda = t^N \prod_{\substack{(a,b),(a,c) \in [\lambda] \\ b < c}} \left(\frac{[h_{ab}]_t}{[h_{ac}]_t} \right)^{\pm \dim S^{\lambda(a,b,c)}}$$

where $N \in \mathbb{Z}$, $h_{xy} = \boldsymbol{\lambda}_x - x + \boldsymbol{\lambda}'_y - y + 1$ is the (x, y)-hook length and $[k]_t = (t^k - 1)/(t - 1)$ is a t-quantum integer.

This is a very beautiful formula but there should be a better formula because the Specht modules are defined over $\mathbb{Z}[t, t^{-1}]$

Gram determinants and the Jantzen sum formula

The Jantzen filtration is a filtration $\underline{S}^\lambda = \underline{S}^\lambda_0 \supseteq \underline{S}^\lambda_1 \supseteq \dots$ where, morally, $\underline{S}^\lambda_k = \{\, x \in \underline{S}^\lambda : p^k | \langle x, y\rangle$ for all $y \in \underline{S}^\lambda \,\}$.



In the Grothendieck group of $\mathcal{H}_n^\Lambda$ we have,

$$\sum_{k>0} [S_k^\lambda] = \sum_{\substack{(a,b),(a,c)\in[\lambda] \\ b<c}} \pm\big(\nu_p([h_{ab}]_t) - \nu_p([h_{ac}]_t)\big)[S^{\lambda(a,b,c)}]$$

Gram determinants and graded Specht modules

If $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$ and $e \in \{0, 2, 3, 4, \dots\}$ define $\deg_e(\boldsymbol{\lambda}) = \sum_{\mathfrak{t}\in\mathrm{Std}(\boldsymbol{\lambda})} \deg_e \mathfrak{t} \quad \in \mathbb{Z}$.

If p is a prime integer define $\mathrm{Deg}_p(\boldsymbol{\lambda}) = \sum_{k\ge1} \deg_{p^k}(\boldsymbol{\lambda}) \quad \in \mathbb{Z}$.

Let $\Phi_e(t)$ be the eth cyclotomic polynomial.

Theorem (Hu-M.)

Suppose that $K = \mathbb{Q}(t)$ and $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. Then

$$\det \underline{G}^\lambda = \begin{cases} t^N \prod_{e>1} \Phi_e(t)^{\deg_e(\boldsymbol{\lambda})}, & \text{if } t \neq 1, \\ \prod_{p \text{ prime}} p^{\mathrm{Deg}_p(\boldsymbol{\lambda})}, & \text{if } t = 1. \end{cases}$$

Consequently, $\deg_e(\boldsymbol{\lambda}) \ge 0$ and $\mathrm{Deg}_p(\boldsymbol{\lambda}) \ge 0$ for all e. Moreover,

Therefore, the semisimple Gram determinant 'knows' about the grading on the cyclotomic quiver Hecke algebra.

The Gelfand-Zetlin subalgebra of $\mathcal{H}_n^\Lambda$

The Gelfand-Zetlin algebra (of type Γ_e) is $\mathcal{L}_n^\Lambda = \langle L_1, L_2, \dots, L_n\rangle$.

Except in semisimple case, we do not have a basis or a presentation for $\mathcal{L}_n^\Lambda$.

If $\mathcal{O} = \mathbb{Q}(t)$ and $\kappa_{l+1} - \kappa_l \ge n$ then:

- $\mathcal{L}_n^\Lambda$ is a split semisimple algebra
- $\mathcal{L}_n^\Lambda$ is a maximal commutative subalgebra of $\mathcal{H}_n^\Lambda$
- $\dim \mathcal{L}_n^\Lambda = \#\,\mathrm{Std}(\mathcal{P}_n^\Lambda)$ and a basis of $\mathcal{L}_n^\Lambda$ is known.

Over an arbitrary field the following hold:

- $\mathcal{L}_n^\Lambda$ is a commutative subalgebra of $\mathcal{H}_n^\Lambda$
- $\mathcal{L}_n^\Lambda$ is split over any field
- The irreducible representations of $\mathcal{L}_n^\Lambda$ and its decomposition matrix are known over any field
- The center of $\mathcal{H}_n^\Lambda$ is $(\mathcal{L}_n^\Lambda)^{\mathfrak{S}_n}$

In the graded setting, $\mathcal{L}_n^\Lambda = \langle y_1, \dots, y_n, e(\mathbf{i}) \mid \mathbf{i} \in I^n\rangle$.
In particular, $\mathcal{L}_n^\Lambda$ is a positively graded subalgebra of $\mathcal{H}_n^\Lambda$.

Seminormal bases

Assume until further notice that $\mathcal{O} = \mathbb{Z}[t, t^{-1}]$, where t is either an indeterminate or 1, and that $\kappa_{l+1} - \kappa_l \ge n$, for $1 \le l \le \ell$.

If $\mathfrak{t} \in \mathrm{Std}(\mathcal{P}_n^\Lambda)$ and $1 \le m \le n$ define the content of m in $\mathfrak{t}$ to be

$$c_m(\mathfrak{t}) = \kappa_l - r + c, \quad \text{if } \mathfrak{t}(l, r, c) = m.$$

Theorem

The Hecke $\mathcal{H}_n^\Lambda$ is semisimple as an $(\mathcal{L}_n^\Lambda, \mathcal{L}_n^\Lambda)$-bimodule with

$$\mathcal{H}_n^\Lambda = \bigoplus_{(\mathfrak{s},\mathfrak{t})\in\mathrm{Std}^2(\mathcal{P}_n^\Lambda)} H_{\mathfrak{s}\mathfrak{t}},$$

where $H_{\mathfrak{s}\mathfrak{t}} = \{\, h \in \mathcal{H}_n^\Lambda : L_m h = [c_m(\mathfrak{s})]h$ and $hL_m = [c_m(\mathfrak{t})]h \,\}$ is a one dimensional $(\mathcal{L}_n^\Lambda, \mathcal{L}_n^\Lambda)$-submodule of $\mathcal{H}_n^\Lambda$.

A seminormal basis of $\mathcal{H}_n^\Lambda$ is any basis $\{f_{\mathfrak{s}\mathfrak{t}}\}$ of $\mathcal{H}_n^\Lambda$ such that

$$f_{\mathfrak{s}\mathfrak{t}}^* = f_{\mathfrak{t}\mathfrak{s}} \quad \text{and} \quad f_{\mathfrak{s}\mathfrak{t}} \in H_{\mathfrak{s}\mathfrak{t}}.$$

Seminormal coefficient systems

Fix a seminormal basis $\{f_{st}\}$ of $\mathcal{H}_n^\Lambda$.

Fact If $s, t \in Std(\mathcal{P}_n^\Lambda)$ then $s = t$ if and only if $c_m(s) = c_m(t)$, $\forall m$

Write $\quad f_{st}T_r = \sum_{(u,v)\in Std^2(\mathcal{P}_n^\Lambda)} a_{uv} f_{uv}$

$\Longrightarrow \quad f_{st}T_r = \sum_{v\in Std(\lambda)} a_{sv} f_{sv}$, by multiplying on the left by L_k

$\Longrightarrow \quad f_{st}T_r = \alpha_r(t) f_{sv} + \beta f_{st}$, where $v = t(r, r+1) = t\cdot s_r$

$\Longrightarrow \quad f_{st}T_r = \alpha_r(t) f_{sv} - \frac{1}{[\rho_r(t)]} f_{st}$, where $\rho_r(t) = c_r(t) - c_{r+1}(t)$.

Using the relation $T_r^2 = (\xi - 1)T_r + \xi$ implies that

$$\alpha_r(t)\alpha_r(v) = \frac{[1-\rho_r(t)][1-\rho_r(v)]}{[\rho_r(t)][\rho_r(v)]}$$

Applying the braid relations,

$$\alpha_r(t)\alpha_{r+1}(t\cdot s_r)\alpha_r(t\cdot s_r s_{r+1}) = \alpha_{r+1}(t)\alpha_r(t\cdot s_{r+1})\alpha_{r+1}(t\cdot s_{r+1}s_r)$$

A seminormal coefficient system is any sets of scalars $\{\alpha_r(t)\}$ which satisfies these last two conditions. Thus, any seminormal basis determines a seminormal coefficient system.

Uniqueness of seminormal bases

Theorem (Seminormal basis theorem)

1. *Every seminormal coefficient system determines a seminormal basis, which is unique up to a choice of scalars.*
2. *If* $s \in Std(\lambda)$ *then* $S^\lambda \cong f_{st}\mathcal{H}_n^\Lambda$ *is irreducible.*
3. *There exist (known) scalars* γ_t *such that* $f_{st}f_{uv} = \delta_{tu}\gamma_t f_{sv}$.
4. *If* $v = t(r, r+1)$ *then* $\gamma_t\alpha_r(v) = \gamma_v\alpha_r(t)$.
5. $\{\, F_t = \frac{1}{\gamma_t} f_{tt} : t \in Std(\lambda), \lambda \in \mathcal{P}_n^\Lambda \,\}$ *is a complete set of pairwise orthogonal primitive idempotents in* $\mathcal{H}_n^\Lambda$ *(or in* $\mathcal{L}_n^\Lambda$*).*

Examples of seminormal coefficient systems:

1. $\alpha_r(t) = \dfrac{[1-\rho_r(t)]}{[\rho_r(t)]}$ whenever $t(r, r+1)$ is standard.
2. $\alpha_r(t) = \begin{cases} 1, & \text{if } t \rhd v = t(r, r+1), \\ \dfrac{[1-\rho_r(t)][1+\rho_r(t)]}{[\rho_r(t)][-\rho_r(t)]}, & \text{if } t \lhd v = t(r, r+1). \end{cases}$

Idempotent subrings

Fix a pair $(\mathcal{K}, t)$ which separates $Std(\mathcal{P}_n^\Lambda)$ where $\mathcal{K}$ is a field and $t \neq 0$.
An idempotent subring is a (local) subring $\mathcal{O}$ of $\mathcal{K}$ such that $t \in \mathcal{O}^\times$ and

1. $c_r(s) \not\equiv c_r(t) \pmod e \quad \Longrightarrow \quad [c_r(s)] - [c_r(t)] \in \mathcal{O}^\times$
2. $\rho_r(s) \not\equiv 0 \pmod e \quad \Longrightarrow \quad [\rho_r(s)] \in \mathcal{O}^\times$
3. $\rho_r(s) \not\equiv 1 \pmod e \quad \Longrightarrow \quad [1-\rho_r(s)] \in \mathcal{O}^\times$

for all $s, t \in Std(\mathcal{P}_n^\Lambda)$ and $1 \le r < n$.

Examples
- Take $\mathcal{K} = \mathbb{Q}$, $t = 1$ and $\mathcal{O} = \mathbb{Z}_{(p)}$, for p prime.
- Take $\mathcal{K} = \mathbb{Q}(\xi, x)$, $t = x + \xi$ and $\mathcal{O} = \mathbb{Q}[x, \xi]_{(x)}$, where x is an indeterminate and $\xi = \exp(2\pi i/e)$.

The point of this definition is the following. For $\mathbf{i} \in I^n$ let

$$Std(\mathbf{i}) = \{\, s \in Std(\mathcal{P}_n^\Lambda) : c_r(s) \equiv i_r \pmod e,\ \text{for } 1 \le r \le n \,\}$$

and define the residue idempotent $f_{\mathbf{i}}^{\mathcal{O}} = \sum_{s\in Std(\mathbf{i})} \frac{1}{\gamma_s} f_{ss} \in \mathcal{L}_n^\Lambda(\mathcal{K})$.

Lemma

Suppose $\mathcal{O}$ *is an idempotent subring and* $\mathbf{i} \in I^n$. *Then* $f_{\mathbf{i}}^{\mathcal{O}} \in \mathcal{L}_n^\Lambda(\mathcal{O})$.

Intertwiners

One of the key KLR relations is that $\psi_r e(\mathbf{i}) = e(s_r\cdot\mathbf{i})\psi_r$.
We want analogous intertwiners for the residue idempotents $f_{\mathbf{i}}^{\mathcal{O}}$.

Lemma

Suppose that $\mathbf{i} \in I^n$ *and* $1 \le r < n$. *Then*

1. *If* $i_r = i_{r+1}$ *then* $T_r f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{i}}^{\mathcal{O}} T_r$.
2. *If* $i_r \neq i_{r+1}$ *then* $(T_rL_r - L_rT_r) f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{j}}^{\mathcal{O}}(T_rL_r - L_rT_r)$, *where* $\mathbf{j} = s_r\cdot\mathbf{i}$.

Proof Compute directly using the seminormal form.
These itertwiners depend on ξ, so we need to 'renormalise' them.
Let $M_r = tL_{r+1} - L_r + 1 \quad \Longrightarrow \quad M_r f_{st} = t^{c_r(s)}[1-\rho_r(s)] f_{st}$.

Lemma

Suppose that $\mathcal{O}$ *is an idempotent subring and* $\mathbf{i} \in I^n$. *Then:*

1. *If* $i_r \neq i_{r+1} + 1$ *then* $\frac{1}{M_r} f_{\mathbf{i}}^{\mathcal{O}} = \sum_{s\in Std(\mathbf{i})} \frac{t^{-c_r(s)}}{[1-\rho_r(s)]} F_s \in \mathcal{H}_n^\Lambda(\mathcal{O})$.
2. *If* $i_r \neq i_{r+1}$ *then* $\frac{1}{L_r - L_{r+1}} f_{\mathbf{i}}^{\mathcal{O}} = \sum_{s\in Std(\mathbf{i})} \frac{t^{c_{r+1}(s)}}{[\rho_r(s)]} F_s \in \mathcal{H}_n^\Lambda(\mathcal{O})$.

Lifting the KLR ψ-generators

For $1 \le r < n$ define $\psi_r^{\mathcal{O}} = \sum_{\mathbf{i}} \psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}$ by

$$\psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (1+T_r)\frac{t^{i_r}}{M_r} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r = i_{r+1}, \\ (T_rL_r - L_rT_r)t^{-i_r} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r = i_{r+1}+1, \\ (T_rL_r - L_rT_r)\frac{1}{M_r} f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise.} \end{cases}$$

By the last slide, $\psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{j}}^{\mathcal{O}} \psi_r^{\mathcal{O}}$ where $\mathbf{j} = s_r \cdot \mathbf{i}$.

We divide by M_r in order to make this elements 'independent' of ξ.

Calculating directly with the seminormal form reveals that

$$\psi_r^{\mathcal{O}} f_{\mathsf{st}} = \delta_{i_r i_{r+1}} \frac{t^{i_r - c_{r+1}(\mathsf{s})}}{[\rho_r(\mathsf{s})]} f_{\mathsf{st}} + \beta_r(\mathsf{s}) f_{\mathsf{ut}},$$

where $\mathsf{u} = \mathsf{s}(r, r+1)$ and $\beta_r(\mathsf{s}) \in \mathcal{K}$ which satisfies the identity

$$\beta_r(\mathsf{s})\beta_r(\mathsf{u}) = \begin{cases} t^{c_r(\mathsf{s})+c_r(\mathsf{u})-i_r-i_{r+1}}[1-\rho_r(\mathsf{s})][1-\rho_r(\mathsf{u})], & \text{if } i_r \leftrightarrows i_{r+1}, \\ t^{c_r(\mathsf{s})-i_r}[1-\rho_r(\mathsf{u})], & \text{if } i_r \to i_{r+1}, \\ t^{c_r(\mathsf{u})-i_{r+1}}[1-\rho_r(\mathsf{s})], & \text{if } i_r \leftarrow i_{r+1}, \\ \frac{t^{2(i_r - c_r(\mathsf{s}))}}{[\rho_r(\mathsf{s})][\rho_r(\mathsf{u})]}, & \text{if } i_r = i_{r+1}, \\ 1, & \text{otherwise.} \end{cases}$$

An integral graded isomorphism theorem

For $1 \le s \le n$ set $y_s^{\mathcal{O}} = \sum_{\mathbf{i} \in I^n} t^{-i_s}(L_s - [i_s]) f_{\mathbf{i}}^{\mathcal{O}}$.

Theorem (Hu-M.)

The algebra $\mathcal{H}_n^\Lambda(\mathcal{O})$ is generated as an $\mathcal{O}$-algebra by the elements

$$\{\, \psi_r^{\mathcal{O}}, y_s^{\mathcal{O}}, f_{\mathbf{i}}^{\mathcal{O}} : 1 \le r < n, 1 \le s \le n, \mathbf{i} \in I^n \,\}$$

subject to the usual KLR relations with deformations of the relations for $y_1^{\mathcal{O}}$ and the quadratic and braid relations which become

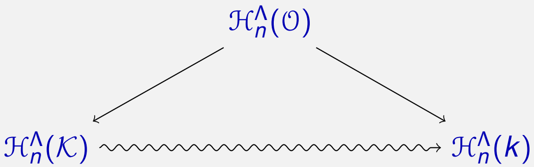
$$(\psi_r^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (\overrightarrow{y}_r^{\mathcal{O}} - y_{r+1}^{\mathcal{O}})(\overleftarrow{y}_{r+1}^{\mathcal{O}} - y_r^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \leftrightarrows i_{r+1}, \\ (\overrightarrow{y}_r^{\mathcal{O}} - y_{r+1}^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \to i_{r+1}, \\ (\overleftarrow{y}_{r+1}^{\mathcal{O}} - y_r^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \leftarrow i_{r+1}, \\ 0, & \text{if } i_r = i_{r+1}, \\ f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise.} \end{cases}$$

and that $(\psi_r^{\mathcal{O}}\psi_{r+1}^{\mathcal{O}}\psi_r^{\mathcal{O}} - \psi_{r+1}^{\mathcal{O}}\psi_r^{\mathcal{O}}\psi_{r+1}^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}} =$

$$= \begin{cases} (\overrightarrow{y}_r^{\mathcal{O}} + \overleftarrow{y}_{r+2}^{\mathcal{O}} - \overrightarrow{y}_{r+1}^{\mathcal{O}} - \overleftarrow{y}_{r+1}^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_{r+2} = i_r \leftrightarrows i_{r+1}, \\ -t^{i_r - i_{r+1}+1} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_{r+2} = i_r \to i_{r+1}, \\ f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_{r+2} = i_r \leftarrow i_{r+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Integral KLR algebras

The KLR algebra now sits inside a 'proper' modular system $(\mathcal{K}, \mathcal{O}, k)$:



The KLR-like presentation of $\mathcal{H}_n^\Lambda(\mathcal{O})$ makes it possible to study the grading on $\mathcal{H}_n^\Lambda$ purely in terms of the seminormal basis.

One application If $\Lambda = \Lambda_0$ then $\mathcal{H}_n^\Lambda = \mathcal{H}_\xi(\mathfrak{S}_n)$ has a subalgebra $\mathcal{H}_\xi(\mathrm{Alt}_n)$, the alternating Hecke algebra. There is a corresponding subalgebra $(\mathcal{R}_n^\Lambda)^{\mathrm{sgn}}$ on the KLR side

Theorem (Boys-M.)

The KLR grading on $\mathcal{H}_\xi(\mathfrak{S}_n)$ restricts to a $\mathbb{Z}$-grading on

$$\mathcal{H}_\xi(\mathit{Alt}_n) \cong (\mathcal{R}_n^\Lambda)^{sgn}.$$

An (almost) graded cellular basis of $\mathcal{H}_n^\Lambda(\mathcal{O})$

Notation: write $\mathsf{s} \blacktriangleright \mathsf{u}$ if $\mathrm{res}(s) = \mathrm{res}(\mathsf{u})$ and $\mathsf{s} \rhd \mathsf{u}$.

Similarly, write $(\mathsf{s}, \mathsf{t}) \blacktriangleright (\mathsf{u}, \mathsf{v})$ if $\mathsf{s} \blacktriangleright \mathsf{u}$, $\mathsf{t} \blacktriangleright \mathsf{v}$ and $(\mathsf{s}, \mathsf{t}) \ne (\mathsf{u}, \mathsf{v})$

So far everything is independent of the choice of seminormal basis. By choosing a particular seminormal basis we can show that there exist elements $\{\, \psi_{\mathsf{st}}^{\mathcal{O}} : (\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\Lambda) \,\}$ such that

- $\psi_{\mathsf{st}}^{\mathcal{O}} \in \mathcal{H}_n^\Lambda(\mathcal{O})$
- $\psi_{\mathsf{st}}^{\mathcal{O}} = f_{\mathsf{st}} + \sum_{(\mathsf{u},\mathsf{v}) \blacktriangleright (\mathsf{s},\mathsf{t})} r_{\mathsf{uv}} f_{\mathsf{uv}}$, for some $r_{\mathsf{uv}} \in \mathcal{K}$
- If $\mathsf{v} = \mathsf{t}(r, r+1)$ and $\mathsf{t} \rhd \mathsf{v}$ then $\psi_{\mathsf{st}}^{\mathcal{O}} \psi_r^{\mathcal{O}} = \psi_{\mathsf{sv}}^{\mathcal{O}}$

Theorem (Hu-M.)

The basis $\{\, \psi_{\mathsf{st}}^{\mathcal{O}} : (\mathsf{s}, \mathsf{t}) \in \mathrm{Std}^2(\mathcal{P}_n^\Lambda) \,\}$ is a cellular basis of $\mathcal{H}_n^\Lambda(\mathcal{O})$.

The $\psi_r^{\mathcal{O}}$-basis gives an $\mathcal{O}$-form of the graded Specht modules. Moreover, $\psi_{\mathsf{st}}^{\mathcal{O}} \otimes_{\mathcal{O}} 1_k$ is a scalar multiple of the graded cellular basis of $\mathcal{H}_n^\Lambda(K) \cong \mathcal{R}_n^\Lambda(K)$, where K is the residue field of $\mathcal{O}$.

Canonical basis for the Graded Specht module?

Suppose now that $\mathcal{O} = K[x,\xi]_{(x)}$ and $t = x + \xi$, for K a field.

Theorem (Hu-M.)

Suppose that $\boldsymbol{\lambda} \in \mathcal{P}_n^\Lambda$. Then $\mathcal{H}_n^\Lambda(\mathcal{O})$ has a unique (cellular) basis $\{\, C^{\mathcal{O}}_{st} : s, t \in \mathrm{Std}^(\mathcal{P}_n^\Lambda)\,\}$ such that

$$C^{\mathcal{O}}_{st} = f_{st} + \sum_{(u,v)\blacktriangleright(s,t)} u_{st} x^{-d_{st}} f_{st}$$

where $u_{st} \in \mathcal{O}^\times$ and $d_{st} \in \mathbb{Z}_{>0}$.

In particular, $C^{\mathcal{O}}_{st}$ depends only on (s,t) and not on the choice of reduced expressions for $d(s)$ and $d(t)$.

Define C_{st} to be the homogeneous component of $C^{\mathcal{O}}_{st} \otimes_{\mathcal{O}} 1_k \in \mathcal{H}_n^\Lambda(k)$ of degree $\deg(s) + \deg(t)$.

Conjecture $C_{st} = C^{\mathcal{O}}_{st} \otimes_{\mathcal{O}} 1_k$ for all $(s,t) \in \mathrm{Std}^2(\mathcal{P}_n^\Lambda)$.

We can probably prove this prove this conjecture but need to check the details.