Ext-algebra of Standard Modules of Rhombal Algebras

for Gradings and Decomposition Numbers Workshop, Stuttgart

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Quasi-hereditary algebras with duality

Let (A, χ) be quasi-hereditary (qh) algebra

Definition (Irving/C.Xi)

We say A is qh with duality (or BGG-algbera) if there is a contravariant functor $\delta : A\text{-mod} \to \text{mod-}A$ which fixes simples.

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Example: δ induced by an anti-automorphism of the algebra.

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A qh algebra is called standard Koszul if all standard module $\Delta(x)$ admits linear projective resolution $\widetilde{\Delta}(x)$:

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1. If A has duality, only need to check condition on Δ
2. Standard Koszul $\Rightarrow$ Koszul (i.e. there is a grading on A with degree 0 part of A admiting linear projective resolution)

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Question: How does A relate to Ext-algebra of X?

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In these cases, A^X is also qh with respect to $(\chi, \leq^{\mathrm{op}})$
Quiver of A^X is the opposite quiver of A
Moreover, A^X is derived equivalent to A

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    - L.P.Li (generalising to...)
2. (particular examples)
    - V.Miemietz-W.Turner [MT] ($GL_2(\overline{\mathbb{F}_p})$)
    - A.Klamt-C.Stroppel [KS] (some generalised Khonvanov arc algebras)

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Second case of [KS]: $\mathfrak{gl}_{n+2}$ with Levi $\mathfrak{gl}_2 \oplus \mathfrak{gl}_n$. Call this algebra K_2^n. So "K_2^n generalises A_n".

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Let B be a weight 2 block of symmetric group and $\overline{B}$ be weight 2 block of Schur algebras, then there is a rhombal algebra U_χ such that

$$\begin{aligned} eU_\chi e &\cong e'Be' \\ f(U_\chi/U_\chi gU_\chi)f &\cong f'(\overline{B}/\overline{B}g'\overline{B})f' \end{aligned}$$

where the unexplained symbols are some sums of primitive idempotents.

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2. Rhombal algebra can relates to K_2^n via truncation, but this truncation satisfy (H).
3. The truncations of principal and RoCK block of weight w, satisfy (H); but not for other weight 2 blocks in general.

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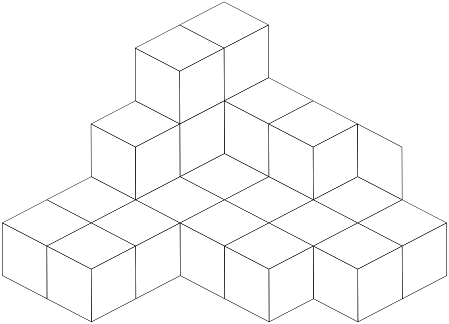
6. And then you impose some relations...

Example

(Part of) An example:



Weight 2 block example

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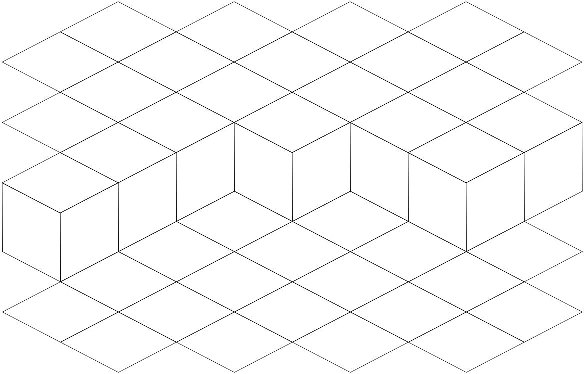

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$\rightsquigarrow$ this can boils down to combinatorics of cubists set (the rhombic tiling)

Homological structure

The partial order on $\mathbb{Z}^3 \supset \chi \quad \rightsquigarrow \quad$ bijection $\lambda : \chi \to \{\text{rhombi}\}$

This gives the standard modules of U_χ:

$$[\Delta(x) : L(y)] = \begin{cases} q^{d(x,y)} & y \leq d(x,y) \leq w = 2 \\ 0 & \text{otherwise} \end{cases}$$

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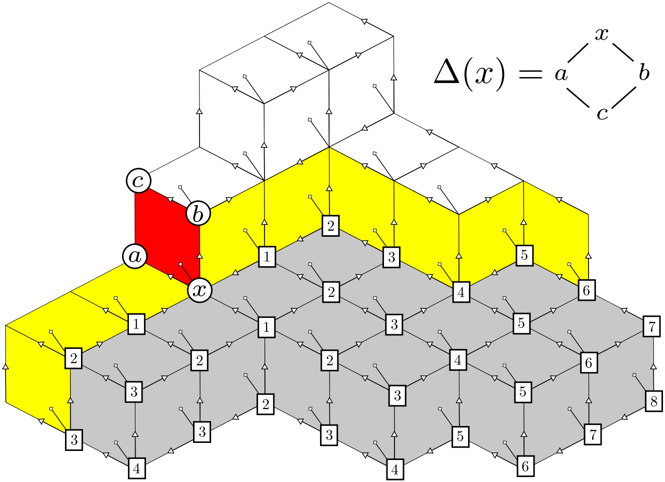
There is also a map $\mu : \chi \to \mathbb{R}^3$ representing $\widetilde{\Delta}(x)$ of $\Delta(x)$ i.e.

$$P(y)\langle i\rangle \text{ a summand of } \widetilde{\Delta}^j(x) \;\Leftrightarrow\; i = j = d(x,y) \text{ and } y \in \mu x$$

Example of visualising $\lambda(x)$ and $\mu(x)$



Combinatorial non-vanishing condition

Theorem

For rhombal algebra U, and $x < y \in \chi$
$\mathrm{Ext}_A(\Delta(x), \Delta(y)) \neq 0$ *precisely when* $\lambda y \cap \mu x \neq \emptyset$ *and for all* $z \in \lambda y \cap \mu x$, $d(x, y) = d(x, z) + d(z, y)$.

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The proof of the theorem is given by looking at the decomposition of the Ext-group into the graded ext-groups

Graded decomposition

If $\mathrm{Ext}^*_U(\Delta(x), \Delta(y))$ non-zero, then it has the following decomposition:

$$\bigoplus_{i=i_0}^{i_0+s} \mathrm{ext}^i_U(\Delta(x), \Delta(y)\langle i - \ (d-i)\ \rangle)$$

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The basis of each graded ext-group is indexed by $z \in \lambda y \cap \mu x$ which are of distance i from x.

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Recall that for Koszul algebras:

$$\mathrm{ext}^i_A(L(x), L(y)\langle j\rangle) \neq 0 \;\;\Rightarrow\;\; i - j = 0$$

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For rhombal algebra which relates with block of symmetric group/Schur algebra, we also calculated all the relations.

Example

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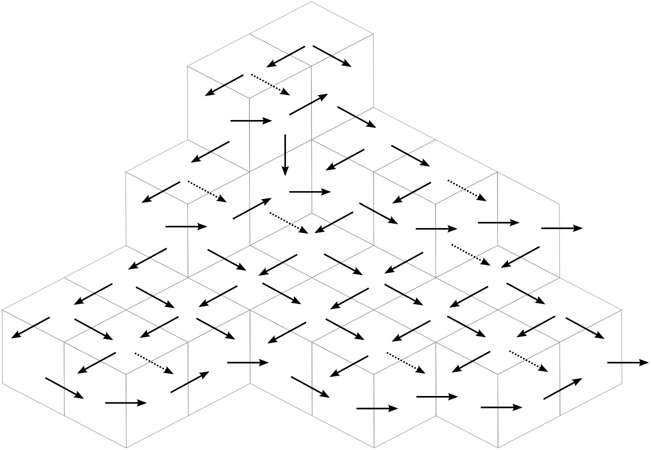

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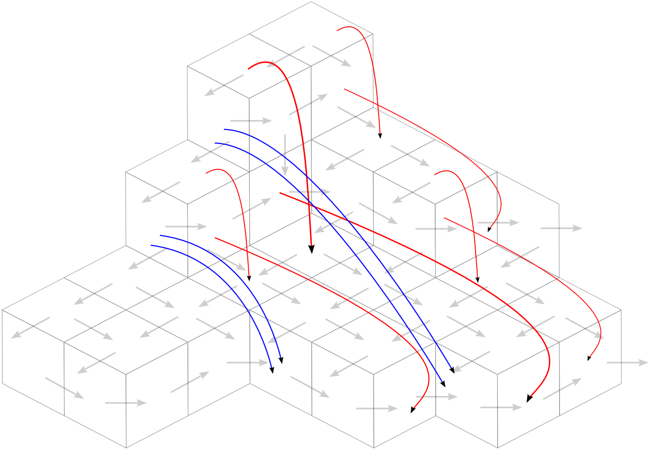

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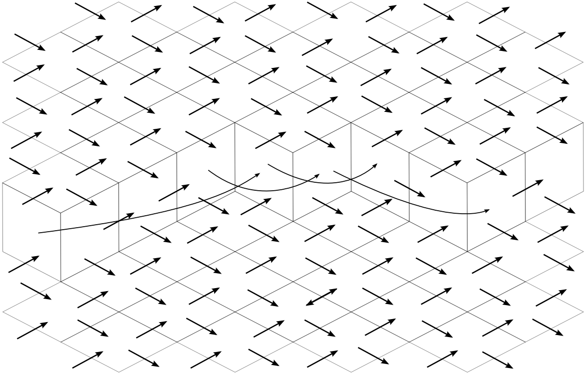

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- Structure of A^{Δ} in general: Quiver of A^{Δ}?
- Homological properties: Formality and derived equivalence?
- How does this help to calculate B^{Δ} for B a weight 2 block of symmetric group/Schur algebra?