Endomorphisms of cell 2-representations

(joint work with Vanessa Miemietz)

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Examples. $\mathscr{S}$ is fiat; $\mathscr{C}_A$ is fiat if and only if A is self-injective and weakly symmetric (i.e. the top and the socle of each indecomposable projective are isomorphic).

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Example. The 2-category $\mathscr{C}_A$ was defined via its **defining** representation.

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Yoneda lemma. For any $\mathbf{M} \in \mathscr{C}\text{-amod}$ we have

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note: $\underbrace{A_j f \otimes_{\Bbbk} eA_i}_{F^*} \otimes_A \underbrace{A_i e \otimes_{\Bbbk} fA_j}_{F} \cong \dim(A_i e)\, A_j f \otimes_{\Bbbk} fA_j$ and $\dim(A_i e)$ is constant on a right cell!!!

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Theorem. $\mathcal{X} := \mathrm{add}\{F\, L_{G_{\mathcal{L}}} : F \in \mathcal{L}\}$ is closed under the action of $\mathscr{C}$

Definition. The **cell 2-representation** $\mathbf{C}_{\mathcal{L}}$ of $\mathscr{C}$ corresponding to $\mathcal{L}$ is the finitary 2-representation obtained by restricting the action of $\mathscr{C}$ to $\mathcal{X}$.

Theorem. Let $\mathcal{J}$ be a 2-sided cell of $\mathscr{C}$ such that:

- different left cells inside $\mathcal{J}$ are not comparable w.r.t. the left order;
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